

Integrable hydrodynamic chains associated with Dorfman Poisson brackets

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Abstract

This paper is devoted to a description of integrable Hamiltonian hydrodynamic chains associated with Dorfman Poisson brackets. Three main classes of these hydrodynamic chains are selected. Generating functions of conservation laws and commuting flows are found. Hierarchies of these Hamiltonian hydrodynamic chains are extended on negative moments and negative time variables. Corresponding three dimensional quasilinear equations of the second order are presented.

Contents

1	Introduction	2
2	Dorfman Poisson brackets and the moment decomposition approach	3
2.1	Negative conservation laws and commuting flows	7
2.2	Generating function of commuting flows and conservation laws	10
2.3	Integrable three dimensional quasilinear equations of the second order . . .	11
3	The first degenerate level $\beta = 0$	15
3.1	Negative conservation laws and commuting flows	18
3.2	Generating function of commuting flows and conservation laws	21
3.3	Integrable three dimensional quasilinear equations of the second order . . .	21
4	The second degenerate level $\beta = 0$ and $\delta = 0$	23
4.1	Negative conservation laws and commuting flows	26
4.2	Generating function of commuting flows and conservation laws	29
4.3	Integrable three dimensional quasilinear equations of the second order . . .	29
5	Conclusion	32
	References	33

1 Introduction

Recently (see [24]), integrable hydrodynamic chains written in the conservative form

$$\partial_t h_k = \partial_x f_k(h_0, h_1, \dots, h_{k+1}), \quad k = 0, 1, 2, \dots \quad (1)$$

were completely described (this classification problem was established in [25], the integrability criterion based on the differential geometric approach was suggested in [9], where first two expressions $f_0(h_0, h_1)$ and $f_2(h_0, h_1, h_2)$ were determined). Also, it was proved that these conservative hydrodynamic chains can be written via special coordinates, i.e. the so called moments $A^k(h_0, h_1, \dots, h_k)$ such that

$$\begin{aligned} A_t^k = & f_1 A_x^{k+1} + f_0 A_x^k + A^{k+1}(s_0 A_x^0 + s_1 A_x^1) + A^k(r_0 A_x^0 + r_1 A_x^1) \\ & + k[A^{k+1}(w_0 A_x^0 + w_1 A_x^1 + w_2 A_x^2) + A^k(v_0 A_x^0 + v_1 A_x^1 + v_2 A_x^2) + A^{k-1}(u_0 A_x^0 + u_1 A_x^1 + u_2 A_x^2)], \end{aligned} \quad (2)$$

where coefficients f_i, s_j, r_k depend on first two moments A^0 and A^1 only, while all other coefficients w_m, v_n, u_p depend just on first three moments A^0, A^1 and A^2 .

A Hamiltonian structure of these integrable hydrodynamic chains is unknown at this moment. Just three Hamiltonian subcases have been found and completely investigated. The Hamiltonian hydrodynamic chains (here $h_{2,k} \equiv \partial h_2 / \partial A^k, k = 0, 1, 2$)

$$A_t^k = 2h_{2,2} A_x^{k+1} + h_{2,1} A_x^k + (k+2)A^{k+1}(h_{2,2})_x + (k+1)A^k(h_{2,1})_x + kA^{k-1}(h_{2,0})_x$$

are associated with the Kupershmidt–Manin Poisson bracket (see the second part in [9], [13] and [19])

$$\{A^k(x), A^n(x')\} = [kA^{k+n-1}D_x + nD_x A^{k+n-1}]\delta(x - x'), \quad k, n = 0, 1, 2, \dots;$$

the Hamiltonian hydrodynamic chains (here $h_{1,k} \equiv \partial h_1 / \partial A^k, k = 0, 1$)

$$A_t^k = (\alpha + \beta)h_{1,1} A_x^{k+1} + \beta h_{1,0} A_x^k + [\alpha(k+1) + 2\beta]A^{k+1}(h_{1,1})_x + (\alpha k + 2\beta)A^k(h_{1,0})_x$$

are associated with the Kupershmidt Poisson brackets (see [8] and [17])

$$\{A^k(x), A^n(x')\} = [(\alpha k + \beta)A^{k+n}D_x + (\alpha n + \beta)D_x A^{k+n}]\delta(x - x'), \quad k, n = 0, 1, 2, \dots;$$

the Hamiltonian hydrodynamic chains (see [18])

$$A_t^k = \beta h_0'(A^0)A_x^{k+1} + (\alpha k + 2\beta)A^{k+1}h_0''(A^0)A_x^0$$

are associated with the simplest Dorfman Poisson bracket (see [5])

$$\{A^k(x), A^n(x')\} = [(\alpha k + \beta)A^{k+n+1}D_x + (\alpha n + \beta)D_x A^{k+n+1}]\delta(x - x'), \quad k, n = 0, 1, 2, \dots$$

If these hydrodynamic chains are integrable, then the corresponding Hamiltonian densities $h_0(A^0)$, $h_1(A^0, A^1)$ and $h_2(A^0, A^1, A^2)$ cannot be arbitrary. The simplest case $h_0(A^0)$

is described in [18] (see also [29]), while a full list of admissible expressions $h_1(A^0, A^1)$ and $h_2(A^0, A^1, A^2)$ is given in [8] and [9], respectively.

Mainly, this paper is devoted to a description of integrable Hamiltonian hydrodynamic chains

$$\begin{aligned} A_t^0 &= 2(\beta A^1 + \delta A^0 + \xi)h_0''(A^0)A_x^0 + h_0'(A^0)(\beta A_x^1 + \delta A_x^0), \\ A_t^k &= h_0'(A^0)(\beta A_x^{k+1} + \delta A_x^k) + [(\alpha k + 2\beta)A^{k+1} + (\gamma k + 2\delta)A^k + \epsilon k A^{k-1}]h_0''(A^0)A_x^0, \quad k = 1, 2, \dots, \end{aligned} \quad (3)$$

associated with more general Dorfman Poisson brackets (see [5])

$$\{A^k(x), A^n(x')\} = [\Gamma^{kn}(\mathbf{A})D_x + D_x\Gamma^{nk}(\mathbf{A})]\delta(x - x'), \quad k, n = 0, 1, 2, \dots \quad (4)$$

where $(\alpha, \beta, \gamma, \delta, \epsilon, \xi)$ are arbitrary constants)

$$\Gamma^{00} = \beta A^1 + \delta A^0 + \xi, \quad \Gamma^{kn} = (\alpha k + \beta)A^{k+n+1} + (\gamma k + \delta)A^{k+n} + \epsilon k A^{k+n-1}, \quad k + n > 0.$$

Hamiltonian densities $h_0(A^0)$ determining integrable hydrodynamic chains (3) are extracted by virtue of the moment decomposition approach (see [29]). The crucial feature of this moment decomposition approach is a reducibility (i.e. the so called infinite dimensional analogue of the Darboux theorem) of Dorfman Poisson brackets to the canonical form d/dx .

This paper is organized in the following way. In Section 2, the first (and most general, i.e. all constants $\alpha, \beta, \gamma, \delta, \epsilon, \xi$ are arbitrary) class of integrable hydrodynamic chains associated with Dorfman Poisson brackets is described. Corresponding Vlasov type kinetic equations are derived. Generating functions of conservation laws and commuting flows are found. A hierarchy of such integrable hydrodynamic chains is extended on negative moments and negative time variables. Three dimensional two component hydrodynamic type systems (as well as three dimensional quasilinear equations of the second order) connected with this hierarchy are presented. In Section 3, the second class (i.e. $\beta = 0$ but $\delta \neq 0$) of integrable hydrodynamic chains is described. In comparison with the general case, these hydrodynamic chains possess a momentum density, but Hamiltonian densities depend on infinitely many moments A^k . In Section 4, the third (and the last, i.e. $\beta = 0$ and $\delta = 0$) class of integrable hydrodynamic chains is described. In Conclusion, a relationship between integrable Hamiltonian hydrodynamic chains and integrable Hamiltonian three dimensional two component hydrodynamic type systems is discussed.

2 Dorfman Poisson brackets and the moment decomposition approach

The above infinite component Dorfman Poisson brackets can be reduced to the canonical form (see [6])

$$\{a^i(x), a^k(x')\} = \frac{\delta_{ik}}{\epsilon_i} \delta'(x - x'), \quad i, k = 1, 2, \dots, N, \quad (5)$$

where ϵ_i are arbitrary constants (δ_{ik} is a Kronecker symbol), due to the so-called “moment decomposition”

$$A^k = \sum f_{k,i}(a^i) \quad (6)$$

with an appropriate choice of functions $f_{k,i}(a^i)$ for *any* natural integer N .

Theorem: Suppose $\beta \neq 0$, in such a general case, Dorfman Poisson bracket (4) reduces to (5) under the moment decomposition

$$dA^k = \beta^{-k} \sum \epsilon_m V(a^m) (V'(a^m) - \delta)^k da^m, \quad k = 0, 1, \dots, \quad (7)$$

where the function $V(p)$ satisfies the ODE

$$VV'' = \frac{\alpha}{\beta} V'^2 + \left(\gamma - \frac{2\alpha\delta}{\beta} \right) V' + \beta\epsilon - \gamma\delta + \frac{\alpha\delta^2}{\beta}, \quad (8)$$

and ξ is an integration constant of the simplest constraint

$$\frac{1}{2} \sum \epsilon_m V^2(a^m) = \beta A^1 + \delta A^0 + \xi. \quad (9)$$

Proof: A substitution of moment decomposition (6) into Poisson bracket (4) implies the recursive relationships

$$(\alpha k + \beta) f'_{k+n+1,i} + (\gamma k + \delta) f'_{k+n,i} + \epsilon k f'_{k+n-1,i} = \frac{1}{\epsilon_i} f'_{n,i} f''_{k,i}, \quad k, n = 0, 1, 2, \dots$$

and constraint (9). For $k = 0$, these ODE's

$$\beta f'_{n+1,i} + \delta f'_{n,i} = \frac{1}{\epsilon_i} f'_{n,i} f''_{0,i}, \quad n = 0, 1, 2, \dots$$

can be reduced to the common form ($\beta \neq 0$)

$$f'_{n,i} = \frac{f'_{0,i}}{\beta^n} \left(\frac{f''_{0,i}}{\epsilon_i} - \delta \right)^n, \quad n = 0, 1, 2, \dots$$

A substitution of these expressions into the above recursive relationships leads to (8), where $V(a^i) = f'_{0,i}/\epsilon_i$. Moreover, taking into account (8), an integration of the differential $d(\alpha A^{k+2} + \gamma A^{k+1} + \epsilon A^k)$ leads (see (7)) to an infinite series of constraints

$$(\alpha k + 2\beta) A^{k+1} + (\gamma k + 2\delta) A^k + \epsilon k A^{k-1} + \xi_k = \beta^{-k} \sum \epsilon_m V^2(a^m) (V'(a^m) - \delta)^k, \quad k = 0, 1, \dots,$$

where ξ_k are integration constants. It is easy to see, if $k = 0$, then $\xi_0 = 2\xi$ (see (9)). The Theorem is proved.

For any positive integer M , an arbitrary Hamiltonian density $h_M(A^0, A^1, \dots, A^M)$ and Poisson bracket (5) determine the Hamiltonian hydrodynamic type system (see [31])

$$a_t^i = \frac{1}{\epsilon_i} \partial_x \frac{\partial h_M}{\partial a^i}, \quad (10)$$

reducible to the symmetric form (see [27]; here $h_{M,m} \equiv \partial h / \partial A^m$, $m = 0, 1, \dots, M$)

$$a_t^i = \frac{1}{\epsilon_i} \partial_x \left(\sum_{m=0}^M h_{M,m} \frac{\partial A^m}{\partial a^i} \right) \equiv \partial_x \left(V(a^i) \sum_{m=0}^M \frac{(V'(a^i) - \delta)^m}{\beta^m} h_{M,m} \right). \quad (11)$$

It is easy to verify that (11) is a hydrodynamic reduction of the Hamiltonian hydrodynamic chain (associated with Dorfman Poisson bracket (4))

$$\begin{aligned}
A_t^0 &= 2\xi(h_{M,0})_x + \sum_{n=0}^M [(\alpha n + 2\beta)A^{n+1} + (\gamma n + 2\delta)A^n + \epsilon n A^{n-1}](h_{M,n})_x \\
&\quad + \sum_{n=0}^M [(\alpha n + \beta)A_x^{n+1} + (\gamma n + \delta)A_x^n + \epsilon n A_x^{n-1}]h_{M,n}, \\
A_t^k &= \sum_{n=0}^M [(\alpha(k+n) + 2\beta)A^{k+n+1} + (\gamma(k+n) + 2\delta)A^{k+n} + \epsilon(k+n)A^{k+n-1}](h_{M,n})_x \\
&\quad + \sum_{n=0}^M [(\alpha n + \beta)A_x^{k+n+1} + (\gamma n + \delta)A_x^{k+n} + \epsilon n A_x^{k+n-1}]h_{M,n}
\end{aligned}$$

by virtue of moment decomposition (7). If the Hamiltonian density $h_M(A^0, A^1, \dots, A^M)$ is an arbitrary function, the above hydrodynamic chain is non-integrable, as well as its hydrodynamic reduction (11). Just in some special cases, this hydrodynamic chain and hydrodynamic reduction (11) became integrable. It means that the Hamiltonian density $h_M(A^0, A^1, \dots, A^M)$ must satisfy some overdetermined system in partial derivatives, which can be obtained utilizing different criteria of an integrability. For instance, in such a case, Haantjes tensor vanishes (see detail in [9]), a family of hydrodynamic reductions (see detail in [7]) parameterizes by N arbitrary functions of a single variable. In this paper, integrable hydrodynamic chains are extracted by the method of symmetric hydrodynamic reductions (see detail in [27], [28], [29]). Without loss of generality, just the simplest Hamiltonian density $h_0(A^0)$ is considered below. Indeed, if the Hamiltonian density $h_0(A^0)$ determines integrable hydrodynamic chain (3), then (3) possesses an infinite series of conservation law densities $h_n(A^0, A^1, \dots, A^n)$; then an infinite series of commuting flows (determined by these Hamiltonian densities $h_n(A^0, A^1, \dots, A^n)$)

$$\begin{aligned}
A_{t^{n+1}}^0 &= 2\xi(h_{n,0})_x + \sum_{m=0}^n [(\alpha m + 2\beta)A^{m+1} + (\gamma m + 2\delta)A^m + \epsilon m A^{m-1}](h_{n,m})_x \\
&\quad + \sum_{m=0}^n [(\alpha m + \beta)A_x^{m+1} + (\gamma m + \delta)A_x^m + \epsilon m A_x^{m-1}]h_{n,m},
\end{aligned} \tag{12}$$

$$\begin{aligned}
A_{t^{n+1}}^k &= \sum_{m=0}^n [(\alpha(k+m) + 2\beta)A^{k+m+1} + (\gamma(k+m) + 2\delta)A^{k+m} + \epsilon(k+m)A^{k+m-1}](h_{n,m})_x \\
&\quad + \sum_{m=0}^n [(\alpha m + \beta)A_x^{k+m+1} + (\gamma m + \delta)A_x^{k+m} + \epsilon m A_x^{k+m-1}]h_{n,m}
\end{aligned}$$

exists. Since $M = 0$, corresponding hydrodynamic reductions (11) simplify to

$$a_{t^1}^i = \partial_x(V(a^i)h'_0). \tag{13}$$

The integrability criterion contains three steps only (see detail in [27], [28], [29] and [22]).

1. Instead (13), let us introduce the generating function of conservation laws for hydrodynamic type system (13) replacing a^i by $p(x, t; \lambda)$, i.e.

$$p_{t^1} = \partial_x(V(p)h'_0), \quad (14)$$

where λ is a free parameter (thus, $a^i = p(x, t, \lambda_i)$, where λ_i are N distinct values of the parameter λ).

2. Under the semi-hodograph transformation $p(x, t; \lambda) \rightarrow \lambda(x, t; p)$, (14) reduces to the Vlasov type kinetic (the collisionless Boltzmann type) equation (see [22])

$$\lambda_{t^1} = V'(p)h'_0\lambda_x - V(p)\lambda_p h''_0 A_x^0. \quad (15)$$

3. Finally, one can check (see [27]) the consistency (15) and corresponding hydrodynamic reduction (13).

Since, all moments A^k are expressed via field variables a^n (see (7)), the last step can be replaced by a verification of consistency (3) and (15). In such a case, all necessary computations (see [8] and [9]) significantly simplify. Indeed, a substitution of $\lambda(p, \mathbf{A})$ into (15) leads to the linear PDE system (here $\lambda_p \equiv \partial\lambda/\partial p$, $\lambda_k \equiv \partial\lambda/\partial A^k$, $k = 0, 1, \dots$)

$$V\lambda_p + \sum_{k=0}^{\infty} [(\alpha k + 2\beta)A^{k+1} + (\gamma k + 2\delta)A^k + \epsilon k A^{k-1}] \lambda_k + \left(2\xi + (\delta - V')\frac{h'_0}{h''_0}\right) \lambda_0 = 0, \quad (16)$$

$$\lambda_k = \frac{\lambda_0}{q^k}, \quad k = 0, 1, 2, \dots, \quad (17)$$

where

$$q = \frac{V'(p) - \delta}{\beta}. \quad (18)$$

A general solution of (17) is given by

$$\lambda(q, \mathbf{A}) = B(q) \sum_{k=0}^{\infty} \frac{A^k}{q^k} + C(q), \quad (19)$$

where $B(q)$ and $C(q)$ are not yet determined functions (“integration constants”). A substitution (19) into (16) leads to an extraction of an *integrable* hydrodynamic chain determined by the Hamiltonian density $h_0(A^0)$ such that (σ is an arbitrary constant; $\alpha \neq 2\beta$)

$$h'_0 = \left(A^0 + \frac{\sigma}{\alpha - 2\beta}\right)^{\frac{\beta}{\alpha - 2\beta}}, \quad (20)$$

while the coefficients $B(q)$ and $C(q)$ in an asymptotic expansion ($q \rightarrow \infty$) are given by their derivatives

$$(\ln B)' = \frac{(\alpha - 2\beta)q^2 - 2\delta q - \epsilon}{q(\alpha q^2 + \gamma q + \epsilon)}, \quad C' = \frac{(\sigma q - 2\xi)B}{\alpha q^2 + \gamma q + \epsilon}.$$

In a particular case $\alpha = 2\beta$, an integrable hydrodynamic chain is determined by the Hamiltonian density $h_0(A^0)$ such that (σ is an arbitrary constant)

$$h'_0 = \exp(A^0/\sigma),$$

while the coefficients $B(q)$ and $C(q)$ in an asymptotic expansion ($q \rightarrow \infty$) are given by their derivatives

$$(\ln B)' = -\frac{2\delta q + \epsilon}{q(2\beta q^2 + \gamma q + \epsilon)}, \quad C' = \frac{(\beta\sigma q - 2\xi)B}{2\beta q^2 + \gamma q + \epsilon}.$$

Equation (8) can be integrated in the parametric form (see (18))

$$V = \exp \int \frac{(\beta q + \delta)dq}{\alpha q^2 + \gamma q + \epsilon}, \quad p = \int \frac{V dq}{\alpha q^2 + \gamma q + \epsilon}. \quad (21)$$

Then all conservation law densities h_k can be found by a substitution of the inverse function $q(\lambda, \mathbf{A})$ (expanded in the Bürmann–Lagrange series, see, for instance, [20]) in (21) at the vicinity $q \rightarrow \infty$.

2.1 Negative conservation laws and commuting flows

All *positive* commuting flows (12) possess infinite sets of conservation laws (cf. (1))

$$\partial_{t^n} h_k = \partial_x f_{n,k}(h_0, h_1, \dots, h_{k+n}), \quad k = 0, 1, 2, \dots, \quad n = 2, 3, \dots$$

In the general case, integrable hydrodynamic chain (3) possesses local conservation laws for positive integers only. It means that any *negative* conservation law density h_{-k} must be expressed via *infinitely many negative* moments A^{-n} . Nevertheless, if $\epsilon = 0$, each negative conservation law density h_{-k} can be expressed via a *finite* number of negative moments A^{-n} . However, $\epsilon = 0$ is not a particular case. Indeed, let us introduce new parameters c_k such that $\alpha = \beta c_2$, $\gamma = c_1 + 2\delta c_2$, $\beta\epsilon = c_0 + c_1\delta + c_2\delta^2$. Thus, instead five arbitrary parameters $\alpha, \beta, \gamma, \delta, \epsilon$, we shall use another five parameters $\beta, \delta, c_0, c_1, c_2$. In such a case, (8) simplifies to (see [28])

$$VV'' = c_2 V'^2 + c_1 V' + c_0. \quad (22)$$

It means that integrable hydrodynamic chain (3) is parameterized by these three essential parameters only, because the parameter δ does not exist in generating function of conservation laws (14) as well as in the Hamiltonian density (see (20))

$$h'_0 = \left(A^0 + \frac{\sigma}{\beta(c_2 - 2)} \right)^{\frac{1}{c_2 - 2}}, \quad (23)$$

while the parameter β can be incorporated to the integration constant σ . It is easy to see below, that the parameter β can be removed from Poisson bracket (4)

$$\Gamma^{kn} = \beta(c_2 k + 1)A^{k+n+1} + (k(c_1 + 2\delta c_2) + \delta)A^{k+n} + \frac{c_0 + c_1\delta + c_2\delta^2}{\beta}kA^{k+n-1}$$

by an appropriate scaling of moments A^n . Thus, aforementioned integrable hydrodynamic chains are determined by one-parametric family of Dorfman Poisson brackets. Since δ is a free parameter, let us choose the special value $\bar{\delta}$ of this parameter such that (i.e. $\epsilon = 0$)

$$c_0 + c_1\bar{\delta} + c_2\bar{\delta}^2 = 0. \quad (24)$$

Then integrable hydrodynamic chain (3)

$$\begin{aligned} A_{t^1}^0 &= 2(\beta A^1 + \bar{\delta} A^0 + \xi)(h'_0)_x + h'_0(\beta A_x^1 + \bar{\delta} A_x^0), \\ A_{t^1}^k &= [\beta(c_2 k + 2)A^{k+1} + (k(c_1 + 2\bar{\delta}c_2) + 2\bar{\delta})A^k](h'_0)_x + h'_0(\beta A_x^{k+1} + \bar{\delta} A_x^k) \end{aligned}$$

possesses an infinite series of negative local conservation laws

$$\partial_{t^1} h_{-k} = \partial_x f_k(h_0, h_{-1}, \dots, h_{-k}), \quad k = 0, 1, 2, \dots$$

The first negative conservation law is given by

$$\partial_{t^1} h_{-1}(A^{-1}) = \bar{\delta} \partial_x [h_{-1}(A^{-1})h'_0(A^0)], \quad (25)$$

where

$$h_{-1} = \left(A^{-1} + \frac{\sigma}{2\bar{\delta}(1-c_2) - c_1} \right)^{\frac{\bar{\delta}}{2\bar{\delta}(1-c_2) - c_1}}$$

Thus, the first *negative* commuting flow

$$\begin{aligned} A_{t^{-1}}^k &= h'_{-1}[\beta(1-c_2)A_x^k + (\bar{\delta}(1-2c_2) - c_1)A_x^{k-1}] \\ &\quad + [\beta(c_2(k-1) + 2)A^k + (k(c_1 + 2\bar{\delta}c_2) + 2\bar{\delta}(1-c_2) - c_1)A^{k-1}](h'_{-1})_x \end{aligned}$$

is determined by the above Hamiltonian density $h_{-1}(A^{-1})$. Then the “zeroth” conservation law is

$$\partial_{t^{-1}} h_0(A^0) = \beta(1-c_2)\partial_x(h'_{-1}(A^{-1})h_0(A^0)). \quad (26)$$

Under moment decomposition (7) extended to negative values of index k , the above hydrodynamic chain transforms to the hydrodynamic reduction

$$a_{t^{-1}}^i = \beta \partial_x \left(\frac{V(a^i)}{V'(a^i) - \bar{\delta}} h'_{-1}(A^{-1}) \right)$$

commuting with hydrodynamic type system (13). A corresponding generating function of conservation laws is given by (cf. (14))

$$p_{t^{-1}} = \beta \partial_x \left(\frac{V(p)}{V'(p) - \bar{\delta}} h'_{-1}(A^{-1}) \right). \quad (27)$$

Thus, obviously, generating function of conservation laws (14) is valid for Hamiltonian densities depended on negative moments too. However, a generating function of *negative*

conservation law densities is associated with *another* expansion (cf. (19)) at the vicinity $q \rightarrow 0$. Indeed, a similar computation as in the previous subsection leads to

$$\lambda(q, \mathbf{A}) = \tilde{B}(q) \sum_{k=-\infty}^{-1} q^{-k-1} A^k + \tilde{C}(q), \quad (28)$$

where

$$(\ln \tilde{B})' = \frac{2\beta(c_2 - 1)q + c_1 + 2\bar{\delta}(c_2 - 1)}{\beta c_2 q^2 + (c_1 + 2\bar{\delta}c_2)q}, \quad \tilde{C}' = -\frac{\sigma \tilde{B}}{\beta c_2 q^2 + (c_1 + 2\bar{\delta}c_2)q}.$$

Thus, a generating function of conservation laws for the integrable hydrodynamic chain determined by the Hamiltonian density $h_{M_1, M_2}(A^{-M_2}, \dots, A^{-1}, A^0, \dots, A^{M_1})$ is given by (cf. (11), (14), (27))

$$p_t = \partial_x \left(V(p) \sum_{m=-M_2}^{M_1} \frac{(V'(p) - \bar{\delta})^m}{\beta^m} h_{M_1, M_2, m} \right),$$

where $h_{M_1, M_2, m} \equiv \partial h_{M_1, M_2} / \partial A^m$, $m = 0, \pm 1, \pm 2, \dots$

Remark: The hierarchy of integrable hydrodynamic chains determined by the Hamiltonian densities $h_{-n}(A^{-n}, \dots, A^{-1})$ and $h_k(A^0, \dots, A^k)$ transforms to itself due to $A^{n-1} \leftrightarrow A^{-n}$ and $t^n \leftrightarrow t^{-n}$. Indeed, let us introduce an auxiliary function $\tilde{V}(p)$ such that (see (27))

$$\tilde{V}(p) = \frac{V(p)}{V'(p) - \bar{\delta}}. \quad (29)$$

This function $\tilde{V}(p)$ satisfies (cf. (22))

$$\tilde{V}\tilde{V}'' = \tilde{c}_2\tilde{V}'^2 + \tilde{c}_1\tilde{V}' + \tilde{c}_0,$$

where

$$\begin{aligned} \tilde{c}_2 &= \frac{2\bar{\delta}c_2 + c_1}{\bar{\delta}(2c_2 - 1) + c_1}, \quad \tilde{c}_1 = 2(c_2 - 1) \frac{2\bar{\delta}c_2 + c_1}{\bar{\delta}(2c_2 - 1) + c_1} - c_2 \\ \tilde{c}_0 &= (c_2 - 1)^2 \frac{2\bar{\delta}c_2 + c_1}{\bar{\delta}(2c_2 - 1) + c_1} + c_2(1 - c_2). \end{aligned}$$

Then an inverse transformation is given by

$$V(p) = [\bar{\delta}(1 - 2c_2) - c_1] \frac{\tilde{V}(p)}{\tilde{V}'(p) + c_2 - 1}. \quad (30)$$

It means, generating function of conservation laws (14) under the transformation $t^1 \leftrightarrow t^{-1}$, $A^0 \leftrightarrow A^{-1}$ reduces to generating function of conservation laws (27). Thus, taking into account (29) and (30), all higher generating functions of conservation laws

$$p_{t^{k+1}} = \partial_x \left(V(p) \sum_{m=0}^k \frac{(V'(p) - \bar{\delta})^m}{\beta^m} h_{k, m} \right), \quad k = 0, 1, 2, \dots \quad (31)$$

transform to corresponding lower generating functions of conservation laws

$$p_{t^{-n}} = \partial_x \left(V(p) \sum_{m=-n}^{-1} \frac{\beta^m}{(V'(p) - \bar{\delta})^m} h_{-n, m} \right), \quad n = 1, 2, \dots \quad (32)$$

and vice versa.

2.2 Generating function of commuting flows and conservation laws

These generating functions (31) and (32) can be incorporated in the sole generating function of conservation laws and commuting flows

$$\partial_{\tau(\zeta)} p(\lambda) = \partial_x G(p(\lambda), p(\zeta)), \quad (33)$$

where the generating function of conservation law densities $p(\lambda)$ is determined above, while an auxiliary function $p(\zeta)$ is obtained from $p(\lambda)$ replacing λ by ζ . Thus, a substitution of an expansion $p(\zeta)$ at the vicinity $q \rightarrow \infty$ or $q \rightarrow 0$ into r.h.s. leads to generating functions (31) and (32) with an appropriate expansion of the so called “vertex” operator $\partial_{\tau(\zeta)}$ with respect to parameter ζ .

Theorem: *The function $G(p(\lambda), p(\zeta))$ is defined by the quadrature*

$$dG(p(\lambda), p(\zeta)) = \left(Q(p(\zeta)) - \frac{V(p(\zeta))R(p(\zeta))}{V'(p(\zeta)) - V'(p(\lambda))} \right) dp(\lambda) + \frac{V(p(\lambda))R(p(\zeta))}{V'(p(\zeta)) - V'(p(\lambda))} dp(\zeta),$$

where

$$R(p(\zeta)) = \frac{V''(p(\zeta))}{V(p(\zeta))}, \quad Q'(p(\zeta)) = (c_2 - 1)R(p(\zeta)). \quad (34)$$

Proof: The compatibility condition $\partial_{t^1}(\partial_{\tau(\zeta)} p(\lambda)) = \partial_{\tau(\zeta)}(\partial_{t^1} p(\lambda))$ (see (14)) implies

$$\partial_{\tau(\zeta)} \ln h'_0 = Q(p(\zeta)) \partial_x \ln h'_0 + R(p(\zeta)) \partial_x p(\zeta), \quad (35)$$

$$\frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\lambda)} = Q(p(\zeta)) - \frac{V(p(\zeta))R(p(\zeta))}{V'(p(\zeta)) - V'(p(\lambda))}, \quad \frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\zeta)} = \frac{V(p(\lambda))R(p(\zeta))}{V'(p(\zeta)) - V'(p(\lambda))}$$

where functions $Q(p(\zeta))$ and $R(p(\zeta))$ are not yet determined. The compatibility condition

$$\frac{\partial}{\partial p(\lambda)} \frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\zeta)} = \frac{\partial}{\partial p(\zeta)} \frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\lambda)}$$

leads to (34). Theorem is proved.

If $c_2 = 1$, then $Q(p(\zeta))$ is a removable constant due to admissible shift in the vertex operator $\partial_{\tau(\zeta)} + \text{const} \partial_x \rightarrow \partial_{\tau(\zeta)}$. Then (35) reduces to the generating function of the “zeroth” conservation laws for all (positive and negative) commuting flows

$$\partial_{\tau(\zeta)} \ln h'_0 = R(p(\zeta)) \partial_x p(\zeta).$$

It is easy to see $h_0 = -\ln h'_0$ in agreement with (23) for $c_2 = 1$.

If $c_2 \neq 1$, then (35) reduces to (see the second equation in (34))

$$\partial_{\tau(\zeta)} [(h'_0)^{c_2-1}] = \partial_x [Q(p(\zeta)) (h'_0)^{c_2-1}]. \quad (36)$$

It is easy to see h_0 satisfies (23) for $c_2 \neq 1$.

2.3 Integrable three dimensional quasilinear equations of the second order

The method of hydrodynamic reductions (see detail in [7]) allows to extract integrable three dimensional quasilinear equations of the second order (see [4]). In this subsection we present a list of some new such equations associated with the hierarchy of commuting hydrodynamic chains described above.

Compatibility conditions $(p_{tk})_{t^n} = (p_{tn})_{t^k}$ lead to integrable three dimensional hydrodynamic type systems. If $k = 1$ and $n = -1$, then the three dimensional hydrodynamic type system (see the general case in [23])

$$u_{t^{-1}} = (1 - c_2)vu_x - uv_x, \quad v_{t^1} = \bar{\delta}uv_x + (c_1 + (2c_2 - 1)\bar{\delta})vu_x$$

is determined by the so called dispersionless Lax pair (see (14) and (27))

$$p_{t^1} = \partial_x (V(p)u), \quad p_{t^{-1}} = \partial_x \left(\frac{V(p)}{V'(p) - \bar{\delta}} v \right), \quad (37)$$

where $u = h'_0(A^0)$ and $v = h'_{-1}(A^{-1})$. If $c_2 \neq 1$ and $\tilde{c}_2 \neq 1$ this hydrodynamic type system possesses two local conservation laws (25) and (26)

$$(u^{c_2-1})_{t^{-1}} = (1 - c_2)(vu^{c_2-1})_x, \quad (v^{\tilde{c}_2-1})_{t^1} = \bar{\delta}(uv^{\tilde{c}_2-1})_x. \quad (38)$$

Then the integrable quasilinear three dimensional equation of the second order (see [4])

$$z_{t^1 t^{-1}} - \frac{z_{t^{-1}}}{z_x} z_{xt^1} = (z_x)^{\frac{1}{c_2-1}} \left(\bar{\delta} z_{xt^{-1}} + \frac{\bar{\delta} c_2 + c_1}{c_2 - 1} \frac{z_{t^{-1}}}{z_x} z_{xx} \right) \quad (39)$$

can be obtain introducing the potential function z such that $z_x = u^{c_2-1}$ and $z_{t^{-1}} = (1 - c_2)vu^{c_2-1}$. A similar equation can be derived from the second conservation law by another potential function \tilde{z} such that $\tilde{z}_x = v^{\tilde{c}_2-1}$ and $\tilde{z}_{t^1} = \bar{\delta}uv^{\tilde{c}_2-1}$. In the particular case $c_2 = 1$, (39) replaces by

$$z_{t^1 t^{-1}} = e^{z_x} [\bar{\delta} z_{xt^{-1}} + (c_1 + (2c_2 - 1)\bar{\delta}) z_{t^{-1}} z_{xx}],$$

while the first conservation law in (38) replaces by $(\ln u)_{t^{-1}} = -v_x$. A substitution (where the potential function ψ is introduced for (37))

$$u = \frac{\psi_{t^1}}{V(\psi_x)}, \quad v = \frac{V'(\psi_x) - \bar{\delta}}{V(\psi_x)} \psi_{t^{-1}}$$

into (38) implies the integrable quasilinear three dimensional equation of the second order

$$\psi_{t^1 t^{-1}} + (c_1 + c_2 \bar{\delta}) \frac{V'(\psi_x) - \bar{\delta}}{V^2(\psi_x)} \psi_{t^1} \psi_{t^{-1}} \psi_{xx} = (1 - c_2) \frac{V'(\psi_x) - \bar{\delta}}{V(\psi_x)} \psi_{t^{-1}} \psi_{xt^1} + \frac{\bar{\delta} \psi_{t^1}}{V(\psi_x)} \psi_{xt^{-1}}.$$

If $k = 1$ and $n = 2$, the dispersionless Lax pair (see (14) and (31) where $M = 1$)

$$p_{t^1} = \partial_x (V(p)u), \quad p_{t^2} = \partial_x (V(p)w + V(p)V'(p)s) \quad (40)$$

determines the three dimensional hydrodynamic type system ($c_2 \neq -1$) written in the conservative form

$$(u^{c_2-1})_{t^1} = \frac{c_2-1}{c_2+1} \left(\frac{w}{u} - \frac{c_1}{c_2} u^{c_2} \right)_x, \quad u_{t^2} = w_{t^1} + \frac{c_0}{c_2+1} (u^{c_2+2})_x, \quad (41)$$

where $u = h'_0(A^0)$, $w = h_{1,0} - \bar{\delta} h_{1,1}$, $s = h_{1,1}$. Moreover, the compatibility condition $(p_{t^1})_{t^2} = (p_{t^2})_{t^1}$ leads to the constraint $s = u^{c_2+1}$, which precisely coincides with a relationship between the corresponding Hamiltonian densities $h_0(A^0)$ and $h_1(A^0, A^1)$, i.e. $h_{1,1} = (h'_0)^{c_2+1}$. Then the quasilinear three dimensional equation of the second order

$$\frac{1}{c_2+1} z_{xt^2} = z_x z_{t^1 t^1} + \left(\frac{z_{t^1}}{c_2-1} + \frac{c_1}{c_2} (z_x)^{\frac{c_2}{c_2-1}} \right) z_{xt^1} + \frac{c_0}{c_2+1} (z_x)^{\frac{c_2+1}{c_2-1}} z_{xx}, \quad (42)$$

is associated with the same (see (39)) potential function z of the first conservation law in (41). In the particular case $c_2 = 1$, (42) replaces by

$$z_{xt^2} = 2z_{t^1 t^1} + 2(z_{t^1} + c_1 e^{z_x}) z_{xt^1} + c_0 e^{2z_x} z_{xx},$$

while the first conservation law in (41) replaces by

$$(\ln u)_{t^1} = \left(\frac{w}{2u} - \frac{c_1 u}{2} \right)_x.$$

A substitution (where the potential function ψ is introduced for (40); see also (37))

$$u = \frac{\psi_{t^1}}{V(\psi_x)}, \quad w = \frac{\psi_{t^2}}{V(\psi_x)} - V'(\psi_x) \left(\frac{\psi_{t^1}}{V(\psi_x)} \right)^{c_2+1}$$

into (41) implies the integrable quasilinear three dimensional equation of the second order

$$\psi_{xt^2} = \left[\frac{\psi_{t^2}}{\psi_{t^1}} + \left(\frac{\psi_{t^1}}{V(\psi_x)} \right)^{c_2} (c_1 - V'(\psi_x)) \right] \psi_{xt^1} + (c_2+1) \left(\frac{\psi_{t^1}}{V(\psi_x)} \right)^{c_2-1} \psi_{t^1 t^1} + c_0 \left(\frac{\psi_{t^1}}{V(\psi_x)} \right)^{c_2+1} \psi_{xx}$$

The most interesting and exceptional case is given by $c_2 = -1$. In such a case, dispersionless Lax pair (40) is no longer correct. The dispersionless Lax pair

$$p_{t^1} = \partial_x (V(p)u), \quad p_{t^2} = \partial_x (V(p)w + V(p)V'(p)s + V(p)F(V'(p))), \quad (43)$$

where $F(q)$ satisfies

$$qF(q) + (-q^2 + c_1 q + c_0)F'(q) = q^2, \quad (44)$$

determines the pair of conservation laws (cf. (41))

$$(u^{-2})_{t^1} + 2 \left(\frac{w + c_1 \ln u + c_1}{u} \right)_x = 0, \quad u_{t^2} = w_{t^1} + c_0 [u(\ln u - 1)]_x. \quad (45)$$

Then the quasilinear three dimensional equation of the second order

$$z_{xt^2} = z_x z_{t^1 t^1} - \left(\frac{z_{t^1}}{2} + c_1 (z_x)^{1/2} \right) z_{xt^1} - \frac{c_0}{2} \ln z_x \cdot z_{xx}$$

follows from the second conservation law of (45), while z is a potential function of the first conservation law of (45).

A comparison (43) with (31) means that the Hamiltonian density h_1 depends on A^0, A^1 (as usual in the general case) and *linearly* on *all* other positive moments A^k (see (12)). A corresponding expression for the function $F(q)$ can be found by a substitution of the Taylor series $F(q) = q^3(\epsilon_3 + \epsilon_4 q + \epsilon_5 q^2 + \dots)$ into (44). Then all coefficients are determined iteratively, i.e.

$$\epsilon_3 = \frac{1}{3c_0}, \quad \epsilon_4 = -\frac{c_1}{4c_0^2}, \quad \epsilon_k = \frac{(k-3)\epsilon_{k-2} - (k-1)c_1\epsilon_{k-1}}{kc_0}, \quad k = 5, 6, \dots$$

The next integrable three dimensional quasilinear equation of the second order follows from (14) and (36)

$$\partial_\tau(u^{c_2-1}) = \partial_x(Q(p)u^{c_2-1}), \quad \partial_{t^1}p = \partial_x(V(p)u).$$

Introducing a potential function z such that (utilizing the first conservation law) $z_x = u^{c_2-1}$, $z_\tau = Q(p)z_x$, the second conservation law transforms to

$$z_x z_{\tau t^1} - z_\tau z_{xt^1} = \frac{1}{c_2 - 1} \frac{V(p(s))}{p'(s)} (z_x)^{\frac{c_2}{c_2-1}} z_{xx} + (z_x)^{\frac{1}{c_2-1}} V'(p(s)) (z_x z_{\tau x} - z_\tau z_{xx}),$$

where $p(s)$ is an inverse function to $Q(p)$ and $s = z_\tau/z_x$. Introducing a potential function ψ such that (utilizing the second conservation law) $\psi_x = p$, $\psi_{t^1} = V(p)u$, the first conservation law transforms to another integrable three dimensional quasilinear equation of the second order

$$\frac{\psi_{t^1\tau}}{\psi_{t^1}} - \frac{V'(\psi_x)}{V(\psi_x)} \psi_{x\tau} = Q(\psi_x) \frac{\psi_{xt^1}}{\psi_{t^1}} + \left(\frac{Q'(\psi_x)}{c_2 - 1} - Q(\psi_x) \frac{V'(\psi_x)}{V(\psi_x)} \right) \psi_{xx}.$$

The most complicated integrable three dimensional quasilinear equation of the second order associated with an integrable hierarchy of commuting hydrodynamic chains presented in this Section (see (31) and (32)) can be obtained utilizing (33). The compatibility condition of two copies of (33)

$$\partial_{\tau^1} p(\lambda) = \partial_x G(p(\lambda), p^1), \quad \partial_{\tau^2} p(\lambda) = \partial_x G(p(\lambda), p^2)$$

leads to the pair of conservation laws

$$\partial_{\tau^1} p^2 = \partial_x G(p^2, p^1), \quad \partial_{\tau^2} p^1 = \partial_x G(p^1, p^2),$$

where $\partial_{\tau^1} = \partial_{\tau(\zeta)}$, $\partial_{\tau^2} = \partial_{\tau(\eta)}$, $p^1 = p(\zeta)$, $p^2 = p(\eta)$ and η is an auxiliary parameter such ζ . Let us introduce two potential functions ψ^1 and ψ^2 such that $\psi_x^1 = p^1$, $\psi_{\tau^1}^1 = G(p^1, p^2)$ and $\psi_x^2 = p^2$, $\psi_{\tau^1}^2 = G(p^2, p^1)$, then the above integrable three dimensional hydrodynamic type system reduces to two equivalent three dimensional quasilinear equations of the second order

$$\partial_{\tau^1} \tilde{G}(\psi_{\tau^2}^1, \psi_x^1) = \partial_x G(\tilde{G}(\psi_{\tau^2}^1, \psi_x^1), \psi_x^1), \quad \partial_{\tau^2} \tilde{G}(\psi_{\tau^1}^2, \psi_x^2) = \partial_x G(\tilde{G}(\psi_{\tau^1}^2, \psi_x^2), \psi_x^2),$$

where $p^1 = \tilde{G}(\psi_{\tau^1}^2, \psi_x^2)$ and $p^2 = \tilde{G}(\psi_{\tau^2}^1, \psi_x^1)$.

Under the transformation of independent variables $x \leftrightarrow t$, generating function of conservation laws (14)

$$p_x = \partial_t(V(p)u)$$

reduces to the form

$$q_t = \partial_x f\left(\frac{q}{u}\right),$$

where $p = f(s)$, $s = V(p)$ and $us = q$. In the particular case $c_0 = 0$, (22) reduces to the form (i.e. $c_2 = c, c_1 = 1$)

$$V(p)V''(p) = cV'^2(p) + V'(p)$$

by an appropriate scaling of the independent variable p . Then the dispersionless Lax pair

$$q_t = \partial_x f\left(\frac{q}{u}\right), \quad q_y = \partial_x f\left(\frac{q}{a}\right), \quad (46)$$

determines the three dimensional hydrodynamic type system

$$a_t = \partial_x f\left(\frac{a}{u}\right), \quad u_y = \partial_x f\left(\frac{u}{a}\right),$$

where

$$f'(s) = \frac{c}{s^c - 1}.$$

Also this three dimensional hydrodynamic type system

$$a_t = \frac{u^{c-2}}{u^c - a^c}(au_x - ua_x), \quad u_y = \frac{a^{c-2}}{u^c - a^c}(au_x - ua_x) \quad (47)$$

can be rewritten as the three dimensional quasilinear equation of the second order (see the general case in [4])

$$Z_{xy} = c^2 \frac{V^{2c}(Z_t)}{(V^c(Z_t) - 1)^2} Z_{xt} + \frac{V^c(Z_t) - 1}{cV(Z_t)} Z_x Z_{yt},$$

where $a = Z_x$ and $u = Z_x/V(Z_t)$. Introducing a potential function ψ such that (see (46)) $q = \psi_x, u = \psi_x/V(\psi_t), a = \psi_x/V(\psi_y)$, (47) reduces to the semi-symmetric form (i.e. this equation is invariant with respect to $y \leftrightarrow t$; see the general case in [4])

$$\psi_{yt} = c \frac{V(\psi_y)V^c(\psi_t)\psi_{xt} - V(\psi_t)V^c(\psi_y)\psi_{xy}}{V^c(\psi_t) - V^c(\psi_y)}.$$

This equation can be written in the form

$$\frac{V^c(\psi_t) - V^c(\psi_y)}{cV(\psi_t)V(\psi_y)}\psi_{yt} = V^{c-1}(\psi_t)\psi_{xt} - V^{c-1}(\psi_y)\psi_{xy}.$$

Let us take two extra copies of this equation

$$\begin{aligned} \frac{V^c(\psi_y) - V^c(\psi_\tau)}{cV(\psi_y)V(\psi_\tau)}\psi_{y\tau} &= V^{c-1}(\psi_y)\psi_{xy} - V^{c-1}(\psi_\tau)\psi_{x\tau}, \\ \frac{V^c(\psi_\tau) - V^c(\psi_t)}{cV(\psi_\tau)V(\psi_t)}\psi_{t\tau} &= V^{c-1}(\psi_\tau)\psi_{x\tau} - V^{c-1}(\psi_t)\psi_{xt}, \end{aligned}$$

where τ is the “fourth time” variable. Eliminating second derivatives $\psi_{x\tau}, \psi_{xt}, \psi_{xy}$, one can obtain the remarkable three dimensional equation of the second order (see [4])

$$\frac{V^c(\psi_t) - V^c(\psi_y)}{V(\psi_t)V(\psi_y)}\psi_{yt} + \frac{V^c(\psi_y) - V^c(\psi_\tau)}{V(\psi_y)V(\psi_\tau)}\psi_{y\tau} + \frac{V^c(\psi_\tau) - V^c(\psi_t)}{V(\psi_\tau)V(\psi_t)}\psi_{t\tau} = 0.$$

3 The first degenerate level $\beta = 0$

In Section 2, the Theorem was formulated for the general case $\beta \neq 0$. This Section is devoted to this degeneration $\beta = 0$.

Theorem: Suppose $\beta = 0$, but $\delta \neq 0$, in such a case, Dorfman Poisson bracket (4) reduces to (5) under the moment decomposition

$$dA^k = \delta \sum \epsilon_m a^m (W'(a^m))^k da^m, \quad k = 0, 1, \dots, \quad (48)$$

where the function $W(p)$ satisfies the ODE

$$\delta p W'' = \alpha W'^2 + \gamma W' + \epsilon, \quad (49)$$

and ξ is an integration constant of the simplest constraint

$$\frac{\delta^2}{2} \sum \epsilon_m (a^m)^2 = \delta A^0 + \xi. \quad (50)$$

Proof: Under the substitution $V = \beta W + \delta p$, (8) reduces to the form

$$\beta W W'' + \delta p W'' = \alpha W'^2 + \gamma W' + \epsilon.$$

Thus, the degenerate case $\beta = 0$ is associated with (49). A substitution of moment decomposition (6) into the Dorfman Poisson bracket (cf. (4))

$$\{A^k(x), A^n(x')\} = [\Gamma^{kn}(\mathbf{A})D_x + D_x \Gamma^{nk}(\mathbf{A})]\delta(x - x'), \quad k, n = 0, 1, 2, \dots$$

where $(\alpha, \gamma, \delta, \epsilon, \xi)$ are arbitrary constants)

$$\Gamma^{00} = \delta A^0 + \xi, \quad \Gamma^{kn} = \alpha k A^{k+n+1} + (\gamma k + \delta) A^{k+n} + \epsilon k A^{k+n-1}, \quad k + n > 0, \quad (51)$$

implies the recursive relationships

$$\alpha n W_{k+n+1} + (\gamma n + \delta) W_{k+n} + \epsilon n W_{k+n-1} = W_k W'_n, \quad k, n = 0, 1, 2, \dots,$$

where $W_n(a^i) = f'_{n,i}/\epsilon_i$. Following the general case (see (7), (8) and below), suppose that a solution of this system is given by $W_m = W_0(W')^m$, where $W_0(p)$ and $W(p)$ are not yet determined. If $n = 0$, then $W_0(p) = \delta p$ (up to an additive constant). A substitution of the above ansatz into the recursive relationships leads to (49). Moreover, taking into account (49), an integration of the differential $d(\alpha A^{k+2} + \gamma A^{k+1} + \epsilon A^k)$ leads (see (48)) to an infinite series of constraints

$$\alpha k A^{k+1} + (\gamma k + 2\delta) A^k + \epsilon k A^{k-1} + \xi_k = \delta^2 \sum \epsilon_m (a^m)^2 (W'(a^m))^k, \quad k = 0, 1, \dots,$$

where ξ_k are integration constants. It is easy to see, if $k = 0$, then $\xi_0 = 2\xi$ (see (50)). The Theorem is proved.

For any positive integer M , an arbitrary Hamiltonian density $h_M(A^0, A^1, \dots, A^M)$ and Dorfman Poisson bracket (51) determine a hydrodynamic chain (see the previous Section), whose Hamiltonian hydrodynamic reduction (10) is presented in the symmetric form (see (48) and [27]; here $h_{M,m} \equiv \partial h / \partial A^m$, $m = 0, 1, \dots, M$)

$$a_t^i = \frac{1}{\epsilon_i} \partial_x \left(\sum_{m=0}^M h_{M,m} \frac{\partial A^m}{\partial a^i} \right) \equiv \delta \partial_x \left(a^i \sum_{m=0}^M (W'(a^i))^m h_{M,m} \right).$$

Then such an integrable hydrodynamic chain possesses the generating function of conservation laws (cf. (31))

$$p_t = \delta \partial_x \left(p \sum_{m=0}^M (W'(p))^m h_{M,m} \right). \quad (52)$$

In this case, the Hamiltonian density satisfies some overdetermined system. Following the general approach presented in the previous Section, we would like restrict our consideration to the simplest case $h_0(A^0)$ only. However, this is impossible. In comparison with full ($\beta \neq 0$) Dorfman Poisson bracket (4), this reduced ($\beta = 0$) Dorfman Poisson bracket leads to the momentum density $P = A^0$, i.e. an arbitrary Hamiltonian hydrodynamic chain possesses a conservation law of the momentum

$$A_t^0 = \left(2\xi h_{M,0} + \alpha \sum_{n=0}^M n h_{M,n} A^{n+1} + \sum_{n=0}^M (\gamma n + 2\delta) h_{M,n} A^n + \epsilon \sum_{n=0}^M n h_{M,n} A^{n-1} - \delta h_M \right)_x.$$

It looks like the simplest integrable case is determined by the next Hamiltonian density $h_1(A^0, A^1)$. However, corresponding hydrodynamic chain

$$A_t^0 = (2\xi h_{1,0} + [\alpha A^2 + (\gamma + 2\delta) A^1 + \epsilon A^0] h_{1,1} + 2\delta A^0 h_{1,0} - \delta h_1)_x,$$

$$A_t^k = [\alpha(k+1)A^{k+2} + (\gamma(k+1) + 2\delta)A^{k+1} + \epsilon(k+1)A^k](h_{1,1})_x$$

+ $[\alpha k A^{k+1} + (\gamma k + 2\delta)A^k + \epsilon k A^{k-1}](h_{1,0})_x + h_{1,1}[\alpha A_x^{k+2} + (\gamma + \delta)A_x^{k+1} + \epsilon A_x^k] + \delta h_{1,0} A_x^k$ depends on the highest moment A^{k+2} . Moreover, any higher commuting flow determined by the Hamiltonian density $h_n(A^0, \dots, A^n)$ depends on the highest moment A^{k+n+1} . Corresponding “time” variable t^{n+1} changes from $n = 1$, while t^0 must be reserved *a priori* for x , because the “zeroth” conservation law density $h_0(A^0)$ is a momentum density A^0 , which cannot create an “intermediate” hydrodynamic chain containing the highest moment A^{k+1} . The momentum density generates just “trivial” commuting flow, i.e. $A_{t^0}^k = A_x^k$. This is a reason to identify x and t^0 . Nevertheless, a commuting flow determined by the “time” variable t^1 exists.

Lemma: *The hydrodynamic chain*

$$A_{t^1}^0 = [2(\xi + \delta A^0)f'' + \delta f' + \epsilon b_1]A_x^0 + [(\gamma + \delta)b_1 + 2\epsilon b_2]A_x^1, \quad (53)$$

$$A_{t^1}^k = [\alpha k A^{k+1} + (\gamma k + 2\delta)A^k + \epsilon k A^{k-1}]f'' A_x^0 + (\delta f' + \epsilon b_1)A_x^k + [(\gamma + \delta)b_1 + 2\epsilon b_2]A_x^{k+1}, \quad k = 1, 2, \dots$$

is determined by the Hamiltonian density depended nonlinearly on the “zeroth” moment and linearly on **all** higher moments, i.e.

$$h_* = f(A^0) + \sum_{n=1}^{\infty} b_n A^n, \quad (54)$$

where b_1 and b_2 are arbitrary constants, while all other constants b_n satisfy the recursive relationships

$$\alpha(n-1)b_{n-1} + (\gamma n + \delta)b_n + \epsilon(n+1)b_{n+1} = 0, \quad n = 2, 3, \dots \quad (55)$$

Proof: can be obtained by a straightforward calculation.

Moreover, integrable hydrodynamic chain (53) is associated with the generating function of conservation laws (see (52))

$$p_{t^1} = \delta \partial_x (p f'(A^0) + \bar{W}(p)), \quad (56)$$

where

$$\bar{W}(p) = p \sum_{m=1}^{\infty} b_m (W'(p))^m. \quad (57)$$

In this case (see [22]),

$$p \bar{W}'' = c_2 \bar{W}'^2 + c_1 \bar{W}' + c_0, \quad (58)$$

where c_k are some constants.

Theorem: Integrable hydrodynamic chain (53) is determined by Hamiltonian density (54), where

$$f''(A^0) = \frac{(\gamma + \delta)b_1 + 2\epsilon b_2}{\alpha A^0 + \sigma[(\gamma + \delta)b_1 + 2\epsilon b_2]}, \quad (59)$$

σ is an integration constant, and constants b_k satisfy (55).

Proof: Under the semi-hodograph transformation $p(\lambda, x, t) \rightarrow \lambda(p, x, t)$, (56) reduces to the Vlasov type kinetic equation (see [22])

$$\lambda_{t^1} = \delta[(f'(A^0) + \bar{W}'(p))\lambda_x - p\lambda_p(f')_x].$$

A substitution $\lambda(p, \mathbf{A})$ leads to

$$\delta p \lambda_p = \left[\frac{(\gamma + \delta)b_1 + 2\epsilon b_2}{f''(A^0)} q - 2\xi - \sum_{k=0}^{\infty} \frac{\alpha k A^{k+1} + (\gamma k + 2\delta)A^k + \epsilon k A^{k-1}}{q^k} \right] \lambda_0 \quad (60)$$

and (17), where (instead (18))

$$q = \frac{\delta \bar{W}'(p) - \epsilon b_1}{(\gamma + \delta)b_1 + 2\epsilon b_2}. \quad (61)$$

Taking into account (see (58) and (61))

$$\frac{\delta^2 p \bar{W}''(p)}{(\gamma + \delta)b_1 + 2\epsilon b_2} = \tilde{c}_2 q^2 + \tilde{c}_1 q + \tilde{c}_0, \quad (62)$$

a substitution (19) into (60) implies the constraints $\tilde{c}_2 = \alpha, \tilde{c}_1 = \gamma, \tilde{c}_0 = \epsilon$, (59) and

$$(\ln B)' = \frac{\alpha q^2 - 2\delta q - \epsilon}{q(\alpha q^2 + \gamma q + \epsilon)}, \quad C' = \frac{\sigma[(\gamma + \delta)b_1 + 2\epsilon b_2]q - 2\xi}{\alpha q^2 + \gamma q + \epsilon} B. \quad (63)$$

Taking into account (61) and the constraints $\tilde{c}_2 = \alpha, \tilde{c}_1 = \gamma, \tilde{c}_0 = \epsilon$, a comparison (49) with (62) implies $q = W'(p)$, i.e.

$$W' = \frac{\delta \bar{W}' - \epsilon b_1}{(\gamma + \delta)b_1 + 2\epsilon b_2}.$$

Differentiating (57) and eliminating \bar{W}' from the above relationship, finally, (56) reduces to

$$p_{t^1} = \partial_x(p(\delta f' + \epsilon b_1) + ((\gamma + \delta)b_1 + 2\epsilon b_2)W), \quad (64)$$

where all constants b_k satisfy (55). Theorem is proved.

Equation (49) can be integrated in the parametric form

$$p = \exp \int \frac{\delta dq}{\alpha q^2 + \gamma q + \epsilon}, \quad W = \int q dp \quad (65)$$

Then all conservation law densities h_k can be found by a substitution of the inverse function $q(\lambda, \mathbf{A})$ (expanded in the B rmann–Lagrange series, see, for instance, [20]) in (65) at the vicinity $q \rightarrow \infty$.

3.1 Negative conservation laws and commuting flows

If hydrodynamic chain (53) is integrable, then an infinite series of higher (positive) commuting flows (55) exist whose corresponding Hamiltonian densities $h_n(A^0, \dots, A^n)$ depend on a finite set of moments only.

If $\epsilon \neq 0$, hydrodynamic chain (53) does not possess negative local conservation laws, i.e. any negative conservation law density h_{-n} depends on all negative moments A^{-k} . However, (49) is reducible to the desirable form

$$\delta p \tilde{W}'' = \alpha \tilde{W}'^2 + (2\alpha c + \gamma) \tilde{W}'$$

due to the shift $W = \tilde{W} + cp$, where the shift constant c is a solution of the quadratic equation $\alpha c^2 + \gamma c + \epsilon = 0$ (cf. (24)). Then reduced Dorfman Poisson bracket (51) transforms to the more simple form (expressed via new moments \tilde{A}^k)

$$\{\tilde{A}^k(x), \tilde{A}^n(x')\} = [\tilde{\Gamma}^{kn}(\tilde{\mathbf{A}})D_x + D_x \tilde{\Gamma}^{nk}(\tilde{\mathbf{A}})]\delta(x - x'), \quad k, n = 0, 1, 2, \dots$$

where

$$\tilde{\Gamma}^{00} = \delta \tilde{A}^0 + \xi, \quad \tilde{\Gamma}^{kn} = \alpha k \tilde{A}^{k+n+1} + [(\gamma + 2\alpha c)k + \delta] \tilde{A}^{k+n}, \quad k + n > 0,$$

and $A^0 = \tilde{A}^0, A^1 = \tilde{A}^1 + c\tilde{A}^0, A^2 = \tilde{A}^2 + 2c\tilde{A}^1 + c^2\tilde{A}^0, A^3 = \tilde{A}^3 + 3c\tilde{A}^2 + 3c^2\tilde{A}^1 + c^3\tilde{A}^0$ etc. This is a consequence following from comparison (see (48))

$$dA^k = \delta \sum \epsilon_m a^m (W'(a^m))^k da^m = \delta \sum \epsilon_m a^m (\tilde{W}'(a^m) + c)^k da^m$$

with

$$d\tilde{A}^k = \delta \sum \epsilon_m a^m (\tilde{W}'(a^m))^k da^m.$$

Thus, without loss of generality, the choice $\epsilon = 0$ allows to simplify all computations in this Section. In such a case, integrable hydrodynamic chain (53)

$$A_{t^1}^0 = [2(\xi + \delta A^0)f'' + \delta f']A_x^0 + (\gamma + \delta)b_1A_x^1, \quad (66)$$

$$A_{t^1}^k = [\alpha k A^{k+1} + (\gamma k + 2\delta)A^k]f''A_x^0 + \delta f'A_x^k + (\gamma + \delta)b_1A_x^{k+1}, \quad k = \pm 1, \pm 2, \dots$$

possesses also an infinite series of lower (negative) local conservation laws. For instance, the first negative conservation law is given by

$$\partial_{t^1} h_{-1}(A^{-1}) = \delta \partial_x [f'(A^0)h_{-1}(A^{-1})],$$

where

$$(\ln h_{-1})' = \frac{\delta}{(2\delta - \gamma)A^{-1} + \sigma(\gamma + \delta)b_1},$$

while (see (59))

$$f''(A^0) = \frac{(\gamma + \delta)b_1}{\alpha A^0 + \sigma(\gamma + \delta)b_1}.$$

Thus, the first negative commuting flow

$$A_{t^{-1}}^k = [\alpha(k-1)A^k + (\gamma(k-1) + 2\delta)A^{k-1}](h'_{-1})_x + h'_{-1}[-\alpha A_x^k + (\delta - \gamma)A_x^{k-1}], \quad k = 0, \pm 1, \pm 2, \dots$$

is determined by the first negative Hamiltonian density $h_{-1}(A^{-1})$. Its first non-negative conservation law (see (54)) is

$$\begin{aligned} \partial_{t^{-1}} h_* &= \partial_x [b_1(\gamma + \delta)A^0 h'_{-1} - \alpha A^0 f' h'_{-1} - \delta f' h_{-1} + (2\delta - \gamma)f' A^{-1} h'_{-1}] \\ &+ \partial_x \left[h'_{-1} \left((\delta - \gamma) \sum_{n=1}^{\infty} b_n A^{n-1} - \alpha \sum_{n=1}^{\infty} b_n A^n \right) \right], \end{aligned}$$

where constants b_n are given by the reduced relationships (see (55))

$$\alpha n b_n + (\gamma(n+1) + \delta)b_{n+1} = 0, \quad n = 1, 2, \dots$$

Under moment decomposition (48) the above hydrodynamic chain transforms to the hydrodynamic reduction

$$a_{t^{-1}}^i = \delta \partial_x \left(\frac{a^i}{W'(a^i)} h'_{-1} \right).$$

Thus, the generating function of conservation laws for the first negative commuting flow

$$p_{t^{-1}} = \delta \partial_x \left(\frac{p}{W'(p)} h'_{-1} \right) \quad (67)$$

is a particular case of (14). Indeed, if $\epsilon = 0$, then ordinary differential equations (22) and (49) are connected by the transformation

$$V(p)W'(p) = p,$$

where (see (21) and (65); b is an integration constant and $b \neq 0$)

$$V(p) = \frac{p}{\gamma} \left(bp^{-\frac{\gamma}{\delta}} - \alpha \right), \quad W(p) = \gamma \int \frac{dp}{bp^{-\frac{\gamma}{\delta}} - \alpha}, \quad (68)$$

and (22) reduces to the particular case

$$VV'' = \frac{\gamma}{\gamma - \delta} \left(V' + \frac{\alpha}{\gamma} \right) \left(V' + \frac{\alpha}{\delta} \right).$$

While a computation of positive conservation laws is based on expansion (19) at the vicinity $q \rightarrow \infty$, negative conservation laws can be found utilizing expansion (28) at the vicinity $q \rightarrow 0$. A generating function of conservation law densities

$$p = \left(\alpha + \frac{\gamma}{q(\lambda, \mathbf{A})} \right)^{-\delta/\gamma}$$

is an inverse expression to $q = W'(p)$ (see (68); without loss of generality, constant b can be fixed to the unity). The first series of higher (positive) conservation law densities can be found by a substitution of an inverse expansion $q(\lambda, \mathbf{A})$ from (19), where (see (63))

$$(\ln B)' = \frac{\alpha q - 2\delta}{q(\alpha q + \gamma)}, \quad C' = \frac{\sigma(\gamma + \delta)b_1 q - 2\xi}{q(\alpha q + \gamma)} B, \quad q \rightarrow \infty.$$

The second series of lower (negative) conservation law densities can be found by a substitution of an inverse expansion $q(\lambda, \mathbf{A})$ from (28), where

$$(\ln \tilde{B})' = \frac{2\alpha q + \gamma - 2\delta}{q(\alpha q + \gamma)}, \quad \tilde{C}' = -\frac{\sigma(\gamma + \delta)b_1}{(\alpha q + \gamma)q} \tilde{B}, \quad q \rightarrow 0. \quad (69)$$

Then substitution of these series of conservation law densities to the generating function of conservation laws (64)

$$p_{t^1} = \partial_x(\delta f' p + (\gamma + \delta)b_1 W) \quad (70)$$

allows to extract both series of conservation laws.

Remark: Functions $\tilde{B}(q)$ and $\tilde{C}(q)$ are found by the same computation as in the previous Section. The Vlasov type kinetic equation

$$\lambda_{t^1} = [\delta f' + (\gamma + \delta)b_1 W'] \lambda_x - \delta p \lambda_p f'' A_x^0$$

is connected with (70) by a semi-hodograph transformation $p(x, t; \lambda) \rightarrow \lambda(x, t; p)$. Taking into account the negative part of integrable hydrodynamic chain (66)

$$A_{t^1}^k = [\alpha k A^{k+1} + (\gamma k + 2\delta) A^k] f'' A_x^0 + \delta f' A_x^k + (\gamma + \delta)b_1 A_x^{k+1}, \quad k = -1, -2, \dots,$$

a substitution of expansion (28) into this Vlasov type kinetic equation leads to (69).

3.2 Generating function of commuting flows and conservation laws

An integrable hierarchy of commuting hydrodynamic chains described in this Section can be embedded into a sole generating function of conservation laws and commuting flows (33). The function $G(p(\lambda), p(\zeta))$ can be found from the compatibility condition of (33) and (70)

$$\partial_{t^1} p(\lambda) = \partial_x [up(\lambda) + W(p(\lambda))], \quad \partial_{\tau(\zeta)} p(\lambda) = \partial_x G(p(\lambda), p(\zeta)).$$

Theorem: *The function $G(p(\lambda), p(\zeta))$ is defined by the quadrature*

$$dG(p(\lambda), p(\zeta)) = \left(Q(p(\zeta)) - \frac{p(\zeta)R(p(\zeta))}{W'(p(\zeta)) - W'(p(\lambda))} \right) dp(\lambda) + \frac{p(\lambda)R(p(\zeta))}{W'(p(\zeta)) - W'(p(\lambda))} dp(\zeta),$$

where

$$R(p(\zeta)) = \frac{\delta W''(p(\zeta))}{p(\zeta)}, \quad Q'(p(\zeta)) = \frac{\alpha}{\delta} R(p(\zeta)). \quad (71)$$

Proof: The compatibility condition $\partial_{t^1}(\partial_{\tau(\zeta)} p(\lambda)) = \partial_{\tau(\zeta)}(\partial_{t^1} p(\lambda))$ implies

$$\partial_{\tau(\zeta)} u = Q(p(\zeta))u_x + R(p(\zeta))\partial_x p(\zeta), \quad (72)$$

$$\frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\lambda)} = Q(p(\zeta)) - \frac{p(\zeta)R(p(\zeta))}{W'(p(\zeta)) - W'(p(\lambda))}, \quad \frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\zeta)} = \frac{p(\lambda)R(p(\zeta))}{W'(p(\zeta)) - W'(p(\lambda))},$$

where functions $Q(p(\zeta))$ and $R(p(\zeta))$ are not yet determined. The compatibility condition

$$\frac{\partial}{\partial p(\lambda)} \frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\zeta)} = \frac{\partial}{\partial p(\zeta)} \frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\lambda)}$$

leads to (71). Theorem is proved.

Moreover (72) reduces to (see the second equation in (71))

$$\partial_{\tau(\zeta)} e^{\alpha u/\delta} = (Q(p(\zeta))e^{\alpha u/\delta})_x$$

It is easy to see the conservation law density $e^{\alpha u/\delta}$ is nothing but a momentum density A^0 (up to unessential additive and multiplicative constants).

3.3 Integrable three dimensional quasilinear equations of the second order

Compatibility conditions $(p_{t^k})_{t^n} = (p_{t^n})_{t^k}$ lead to integrable three dimensional hydrodynamic type systems. If $k = 1$ and $n = -1$, then three dimensional hydrodynamic type system

$$u_{t^{-1}} + v_{t^0} + \frac{\alpha}{\delta} v u_{t^0} = 0, \quad v_{t^1} = u v_{t^0} + \frac{\gamma - \delta}{\delta} v u_{t^0} \quad (73)$$

is determined by the dispersionless Lax pair (see (67) and (70))

$$p_{t^1} = \partial_{t^0}(up + W(p)), \quad p_{t^{-1}} = \partial_{t^0} \left(\frac{p}{W'(p)} v \right), \quad (74)$$

where

$$u = \frac{\delta}{(\gamma + \delta)b_1} f', \quad v = \delta h'_{-1}.$$

Three dimensional hydrodynamic type system (73) can be written in the conservative form ($\gamma \neq \delta$)

$$(e^{\alpha u/\delta})_{t^{-1}} + \frac{\alpha}{\delta} (v e^{\alpha u/\delta})_{t^0} = 0, \quad (v^{\frac{\delta}{\gamma-\delta}})_{t^1} = (u v^{\frac{\delta}{\gamma-\delta}})_{t^0}.$$

Introducing the potential function z such that (see the first conservation law)

$$u = \frac{\delta}{\alpha} \ln z_{t^0}, \quad v = -\frac{\delta z_{t^{-1}}}{\alpha z_{t^0}},$$

(73) reduces to the three dimensional quasilinear equation of the second order

$$\alpha(z_{t^0} z_{t^{-1}t^1} - z_{t^{-1}} z_{t^0t^1}) = \delta z_{t^0} \ln z_{t^0} \cdot z_{t^{-1}t^0} + (\gamma - \delta - \delta \ln z_{t^0}) z_{t^{-1}} z_{t^0t^0};$$

introducing the potential function \tilde{z} such that (see the second conservation law)

$$v = (\tilde{z}_{t^0})^{\frac{\gamma-\delta}{\delta}}, \quad u = \frac{\tilde{z}_{t^1}}{\tilde{z}_{t^0}},$$

(73) reduces to another three dimensional quasilinear equation of the second order

$$\delta(\tilde{z}_{t^0} \tilde{z}_{t^1t^{-1}} - \tilde{z}_{t^1} \tilde{z}_{t^0t^{-1}}) + (\tilde{z}_{t^0})^{\frac{\gamma-\delta}{\delta}} [(\gamma - \delta) \tilde{z}_{t^0} - \alpha \tilde{z}_{t^1}] \tilde{z}_{t^0t^0} + \alpha (\tilde{z}_{t^0})^{\frac{\gamma-\delta}{\delta}+1} \tilde{z}_{t^0t^1} = 0.$$

A substitution (where the potential function ψ is introduced for (74))

$$u = \frac{\psi_{t^1} - W(\psi_{t^0})}{\psi_{t^0}}, \quad v = \frac{W'(\psi_{t^0})}{\psi_{t^0}} \psi_{t^{-1}}$$

into (73) implies the integrable quasilinear three dimensional equation of the second order

$$\begin{aligned} & \psi_{t^{-1}t^1} + \frac{\alpha W'(\psi_{t^0})}{\delta \psi_{t^0}} \psi_{t^{-1}} \psi_{t^0t^1} - \frac{\psi_{t^1} - W(\psi_{t^0})}{\psi_{t^0}} \psi_{t^0t^{-1}} \\ & + \frac{W'(\psi_{t^0})}{\delta \psi_{t^0}^2} (\alpha W(\psi_{t^0}) + (\gamma - \delta) \psi_{t^0} - \alpha \psi_{t^1}) \psi_{t^{-1}} \psi_{t^0t^0} = 0. \end{aligned}$$

If $k = 1$ and $n = 2$, then the dispersionless Lax pair (see (52))

$$p_{t^1} = \partial_{t^0}(up + W(p)), \quad p_{t^2} = \partial_{t^0}(wp + spW'(p)) \quad (75)$$

determines the constraint $s = e^{\alpha u/\delta}$ and the three dimensional hydrodynamic type system

$$(e^{\alpha u/\delta})_{t^1} = \left(w + \left(u - \frac{2\delta + \gamma}{\alpha} \right) e^{\alpha u/\delta} \right)_{t^0}, \quad u_{t^2} + uw_{t^0} = w_{t^1} + wu_{t^0}, \quad (76)$$

which is equivalent to the three dimensional quasilinear equation of the second order

$$\alpha \delta z_{t^0t^2} = \alpha^2 z_{t^0} z_{t^1t^1} + \alpha(\gamma + \delta - 2\delta \ln z_{t^0}) z_{t^0} z_{t^0t^1} + \delta[\delta z_{t^0} (\ln z_{t^0})^2 - (\gamma + 2\delta) z_{t^0} (\ln z_{t^0} - 1) + \alpha z_{t^1}] z_{t^0t^0},$$

where z is a potential function for the conservation law in (76). A substitution (where the potential function ψ is introduced for (75))

$$u = \frac{\psi_{t^1} - W(\psi_{t^0})}{\psi_{t^0}}, \quad w = \frac{\psi_{t^2}}{\psi_{t^0}} - W'(\psi_{t^0}) \exp\left(\frac{\alpha\psi_{t^1} - \alpha W(\psi_{t^0})}{\delta\psi_{t^0}}\right)$$

into (76) implies the integrable quasilinear three dimensional equation of the second order

$$\begin{aligned} \delta \exp\left(\frac{\alpha W(\psi_{t^0}) - \alpha\psi_{t^1}}{\delta\psi_{t^0}}\right) \cdot \left(\psi_{t^0 t^2} - \frac{\psi_{t^2}}{\psi_{t^0}} \psi_{t^0 t^0}\right) &= \alpha\psi_{t^1 t^1} + \left(\gamma + \delta - 2\alpha \frac{\psi_{t^1} - W(\psi_{t^0})}{\psi_{t^0}}\right) \psi_{t^0 t^1} \\ &+ \left(\alpha \frac{[\psi_{t^1} - W(\psi_{t^0})]^2}{\psi_{t^0}^2} - (\gamma + \delta) \frac{\psi_{t^1} - W(\psi_{t^0})}{\psi_{t^0}} - \delta W'(\psi_{t^0})\right) \psi_{t^0 t^0}. \end{aligned}$$

The next integrable three dimensional quasilinear equation of the second order follows from (70) and (72)

$$\partial_\tau e^{\alpha u/\delta} = \partial_{t^0}(Q(p)e^{\alpha u/\delta}), \quad \partial_{t^1} p = \partial_{t^0}(up + W(p)).$$

Introducing a potential function z such that (utilizing the first conservation law) $z_{t^0} = e^{\alpha u/\delta}$, $z_\tau = Q(p)z_{t^0}$, the second conservation law transforms to

$$z_{t^0} z_{\tau t^1} - z_\tau z_{t^0 t^1} = \frac{\delta p(s)}{\alpha p'(s)} z_{t^0} z_{t^0 t^0} + \left(\frac{\delta}{\alpha} \ln z_{t^0} + W'(p(s))\right) (z_{t^0} z_{\tau t^0} - z_\tau z_{t^0 t^0}),$$

where $p(s)$ is an inverse function to $Q(p)$ and $s = z_\tau/z_{t^0}$. Introducing a potential function ψ such that (utilizing the second conservation law) $\psi_{t^0} = p$, $\psi_{t^1} = up + W(p)$, the first conservation law transforms to another integrable three dimensional quasilinear equation of the second order

$$\begin{aligned} \psi_{t^0} \psi_{t^1 \tau} &= (\psi_{t^1} \psi_{t^0 \tau} + \psi_{t^0} W'(\psi_{t^0}) - W(\psi_{t^0})) \psi_{t^0 \tau} + \psi_{t^0} Q(\psi_{t^0}) \psi_{t^0 t^1} \\ &+ \left(Q(\psi_{t^0})(W(\psi_{t^0}) - \psi_{t^0} W'(\psi_{t^0}) - \psi_{t^1}) + \frac{\delta}{\alpha} \psi_{t^0}^2 Q'(\psi_{t^0})\right) \psi_{t^0 t^0}. \end{aligned}$$

4 The second degenerate level $\beta = 0$ and $\delta = 0$

In Section 3, the Theorem was formulated for the case $\beta = 0$ but $\delta \neq 0$. This Section is devoted to the second degeneration $\delta = 0$.

Theorem: Suppose $\beta = 0$ and $\delta = 0$, in such a case, Dorfman Poisson bracket (4) reduces to (5) under the moment decomposition

$$dA^k = \sum \epsilon_m (U'(a^m))^k da^m, \quad k = 0, 1, \dots, \quad (77)$$

where the function $U(p)$ satisfies the ODE

$$U'' = \alpha U'^2 + \gamma U' + \epsilon, \quad (78)$$

and ξ is an integration constant of the simplest constraint

$$\sum \epsilon_m = \xi. \quad (79)$$

Proof: Under the substitution $V = \beta U + \delta p + 1$, (8) reduces to the form (cf. (49))

$$\beta U U'' + (\delta p + 1) U'' = \alpha U'^2 + \gamma U' + \epsilon$$

Thus, the double degenerate case $\beta = 0$ and then $\delta = 0$ is associated with (78). A substitution of moment decomposition (6) into the Dorfman Poisson bracket (cf. (4))

$$\{A^k(x), A^n(x')\} = [\Gamma^{kn}(\mathbf{A}) D_x + D_x \Gamma^{nk}(\mathbf{A})] \delta(x - x'), \quad k, n = 0, 1, 2, \dots \quad (80)$$

where $(\alpha, \gamma, \epsilon, \xi)$ are arbitrary constants)

$$\Gamma^{00} = \xi, \quad \Gamma^{kn} = k(\alpha A^{k+n+1} + \gamma A^{k+n} + \epsilon A^{k+n-1}), \quad k + n > 0,$$

implies the recursive relationships

$$n(\alpha U_{k+n+1} + \gamma U_{k+n} + \epsilon U_{k+n-1}) = U_k U'_n, \quad k, n = 0, 1, 2, \dots,$$

where $U_n(a^i) = f'_{n,i}/\epsilon_i$. Following the general case (see (7), (8) and below), suppose that a solution of this system is given by $U_m = U_0(U')^m$, where $U_0(p)$ and $U(p)$ are not yet determined. If $n = 0$, then $U_0(p) = 1$ (an arbitrary nonzero additive constant is fixed to the unity here). A substitution of the above ansatz into the recursive relationships leads to (78). Moreover, taking into account (78), an integration of the differential $d(\alpha A^{k+2} + \gamma A^{k+1} + \epsilon A^k)$ leads (see (77)) to an infinite series of constraints

$$k(\alpha A^{k+1} + \gamma A^k + \epsilon A^{k-1}) + \xi_k = \sum \epsilon_m (U'(a^m))^k, \quad k = 0, 1, \dots,$$

where ξ_k are integration constants. It is easy to see, if $k = 0$, then $\xi_0 = \xi$ (see (79)). The Theorem is proved.

For any positive integer M , an arbitrary Hamiltonian density $h_M(A^0, \dots, A^M)$ and Dorfman Poisson bracket (80) determine a hydrodynamic chain (see the previous Sections), whose Hamiltonian hydrodynamic reduction (10) is presented in the symmetric form (see (77) and [27]; here $h_{M,m} \equiv \partial h / \partial A^m, m = 0, 1, \dots, M$)

$$a_t^i = \frac{1}{\epsilon_i} \partial_x \left(\sum_{m=0}^M h_{M,m} \frac{\partial A^m}{\partial a^i} \right) \equiv \partial_x \left(\sum_{m=0}^M (U'(a^i))^m h_{M,m} \right).$$

Then such an integrable hydrodynamic chain possesses the generating function of conservation laws (cf. (31))

$$p_t = \partial_x \left(\sum_{m=0}^M (U'(p))^m h_{M,m} \right). \quad (81)$$

In this case, the Hamiltonian density satisfies some overdetermined system. Since this double degenerate case ($\beta = 0$ and $\delta = 0$) is very familiar to the degenerate case ($\beta = 0$), very similar results are presented below.

Lemma: *The hydrodynamic chain*

$$A_{t^1}^0 = (2\xi f' + \epsilon b_1 A^0 + (\gamma b_1 + 2\epsilon b_2) A^1)_x, \quad (82)$$

$$A_{t^1}^k = k(\alpha A^{k+1} + \gamma A^k + \epsilon A^{k-1}) f'' A_x^0 + \epsilon b_1 A_x^k + (\gamma b_1 + 2\epsilon b_2) A_x^{k+1}, \quad k = 1, 2, \dots$$

is determined by Hamiltonian density (54) where b_1 and b_2 are arbitrary constants, while all other constants b_n satisfy the recursive relationships

$$\alpha(n-1)b_{n-1} + \gamma n b_n + \epsilon(n+1)b_{n+1} = 0, \quad n = 2, 3, \dots \quad (83)$$

Proof: can obtained by a straightforward calculation.

Moreover, integrable hydrodynamic chain (82) is associated with the generating function of conservation laws (see (81))

$$p_{t^1} = \partial_x(f'(A^0) + \bar{U}(p)), \quad (84)$$

where

$$\bar{U}(p) = \sum_{m=1}^{\infty} b_m (U'(p))^m. \quad (85)$$

In this case (see [28]),

$$\bar{U}'' = c_2 \bar{U}'^2 + c_1 \bar{U}' + c_0, \quad (86)$$

where c_k are some constants.

Theorem: *Integrable hydrodynamic chain (82) is determined by Hamiltonian density (54), where*

$$f''(A^0) = \frac{\gamma b_1 + 2\epsilon b_2}{\sigma + \alpha A^0}, \quad (87)$$

σ is an integration constant and b_k satisfy (83).

Proof: Under the semi-hodograph transformation $p(x, t; \lambda) \rightarrow \lambda(x, t; p)$, (84) reduces to the Vlasov type kinetic equation (see [22])

$$\lambda_{t^1} = \bar{U}'(p) \lambda_x - \lambda_p f''(A^0) A_x^0.$$

A substitution $\lambda(p, \mathbf{A})$ leads to

$$\lambda_p = \left(\frac{\bar{U}'(p) - \epsilon b_1}{f''(A^0)} - 2\xi - \sum_{k=1}^{\infty} \frac{k(\alpha A^{k+1} + \gamma A^k + \epsilon A^{k-1})}{q^k} \right) \lambda_0 \quad (88)$$

and (17), where (instead (18) and (61))

$$q = \frac{\bar{U}'(p) - \epsilon b_1}{\gamma b_1 + 2\epsilon b_2}. \quad (89)$$

Taking into account (see (86) and (89))

$$\frac{\bar{U}''}{\gamma b_1 + 2\epsilon b_2} = \tilde{c}_2 q^2 + \tilde{c}_1 q + \tilde{c}_0, \quad (90)$$

a substitution (19) into (88) implies the constraints $\tilde{c}_2 = \alpha, \tilde{c}_1 = \gamma, \tilde{c}_0 = \epsilon$, (87) and

$$(\ln B)' = \frac{\alpha q^2 - \epsilon}{q(\alpha q^2 + \gamma q + \epsilon)}, \quad C' = \frac{\sigma q - 2\xi}{\alpha q^2 + \gamma q + \epsilon} B. \quad (91)$$

Taking into account (89) and the constraints $\tilde{c}_2 = \alpha, \tilde{c}_1 = \gamma, \tilde{c}_0 = \epsilon$, a comparison (78) with (90) implies $q = U'(p)$, i.e.

$$U' = \frac{\bar{U}' - \epsilon b_1}{\gamma b_1 + 2\epsilon b_2}.$$

Differentiating (85) and eliminating \bar{U}' from the above relationship, finally, (84) reduces to

$$p_{t^1} = \partial_x(f'(A^0) + (\gamma b_1 + 2\epsilon b_2)U(p) + \epsilon b_1 p), \quad (92)$$

where constants b_k satisfy (83). Theorem is proved.

Equation (78) can be integrated in the parametric form

$$p = \int \frac{dq}{\alpha q^2 + \gamma q + \epsilon}, \quad U = \int \frac{q dq}{\alpha q^2 + \gamma q + \epsilon}. \quad (93)$$

Then all conservation law densities h_k can be found by a substitution of the inverse function $q(\lambda, \mathbf{A})$ (expanded in the B rmann–Lagrange series, see, for instance, [20]) in (93) at the vicinity $q \rightarrow \infty$.

4.1 Negative conservation laws and commuting flows

If hydrodynamic chain (82) is integrable, then an infinite series of higher (positive) commuting flows exist whose corresponding Hamiltonian densities $h_n(A^0, \dots, A^n)$ depend on a finite set of moments only.

If $\epsilon \neq 0$, hydrodynamic chain (82) does not possess negative local conservation laws, i.e. any negative conservation law density h_{-n} depends on all negative moments A^{-k} . However, (78) is reducible to the desirable form

$$\tilde{U}'' = \alpha \tilde{U}'^2 + (\gamma + 2\alpha c)\tilde{U}'$$

due to the shift $U = \tilde{U} + cp$, where the shift constant c is a solution of the quadratic equation $\alpha c^2 + \gamma c + \epsilon = 0$ (cf. (24)). Then reduced Dorfman Poisson bracket (80) transforms to the more simple form (expressed via new moments \tilde{A}^k)

$$\{\tilde{A}^k(x), \tilde{A}^n(x')\} = [\tilde{\Gamma}^{kn}(\tilde{\mathbf{A}})D_x + D_x \tilde{\Gamma}^{nk}(\tilde{\mathbf{A}})]\delta(x - x'), \quad k, n = 0, 1, 2, \dots$$

where

$$\tilde{\Gamma}^{00} = \xi, \quad \tilde{\Gamma}^{kn} = k(\alpha \tilde{A}^{k+n+1} + (\gamma + 2\alpha c)\tilde{A}^{k+n}), \quad k + n > 0,$$

and $A^0 = \tilde{A}^0, A^1 = \tilde{A}^1 + c\tilde{A}^0, A^2 = \tilde{A}^2 + 2c\tilde{A}^1 + c^2\tilde{A}^0, A^3 = \tilde{A}^3 + 3c\tilde{A}^2 + 3c^2\tilde{A}^1 + c^3\tilde{A}^0$ etc. This is a consequence following from comparison (see (77))

$$dA^k = \sum \epsilon_m (U'(a^m))^k da^m = \sum \epsilon_m (\tilde{U}'(a^m) + c)^k da^m$$

with

$$d\tilde{A}^k = \sum \epsilon_m (\tilde{U}'(a^m))^k da^m.$$

Thus, without loss of generality, the choice $\epsilon = 0$ allows to simplify all computations in this Section. In such a case, integrable hydrodynamic chain (82)

$$A_{t^1}^0 = (2\xi f' + \gamma b_1 A^1)_x, \quad A_{t^1}^k = k(\alpha A^{k+1} + \gamma A^k) f'' A_x^0 + \gamma b_1 A_x^{k+1}, \quad k = \pm 1, \pm 2, \dots \quad (94)$$

possesses also an infinite series of lower (negative) local conservation laws. For instance, the first negative conservation law is given by ($\gamma \neq 0$)

$$\partial_{t^1} \ln(\gamma A^{-1} - \sigma) = -\frac{\gamma^2 b_1}{\alpha} \partial_x \ln(\alpha A^0 + \sigma) \quad (95)$$

where (see (87))

$$f'(A^0) = \frac{\gamma b_1}{\alpha} \ln(\alpha A^0 + \sigma).$$

Thus, the first negative commuting flow

$$A_{t^{-1}}^k = (k-1)(\alpha A^k + \gamma A^{k-1})(h'_{-1})_x - h'_{-1}(\alpha A_x^k + \gamma A_x^{k-1})$$

is determined by the first negative Hamiltonian density $h_{-1}(A^{-1}) = \ln(\gamma A^{-1} - \sigma)$. Its first non-negative conservation law (see (54)) is

$$\partial_{t^{-1}} h_* = \left(\frac{\gamma^2}{\alpha} b_1 \ln(\sigma + \alpha A^0) + \gamma b_1 A^0 h'_{-1} - (\alpha A^0 + \gamma A^{-1}) f' h'_{-1} - \sum_{n=1}^{\infty} b_n (\alpha A^n + \gamma A^{n-1}) h'_{-1} \right)_x$$

where constants b_n are given by the reduced relationships (see (83))

$$\alpha n b_n + \gamma(n+1) b_{n+1} = 0, \quad n = 1, 2, \dots$$

Under moment decomposition (77) the above hydrodynamic chain transforms to the hydrodynamic reduction

$$a_{t^{-1}}^i = \partial_x \frac{h'_{-1}}{U'(a^i)}.$$

Thus, the generating function of conservation laws for the first negative commuting flow

$$p_{t^{-1}} = \partial_x \frac{h'_{-1}}{U'(p)}. \quad (96)$$

is a particular case of (14). Indeed, if $\epsilon = 0$ and $\delta = 0$, then ordinary differential equations (22) and (78) are connected by the transformation

$$V(p) U'(p) = 1,$$

where (see (21) and (93))

$$V(p) = \frac{e^{-\gamma p} - \alpha}{\gamma}, \quad U(p) = \frac{1}{\alpha} \ln \gamma - \frac{1}{\alpha} \ln(1 - \alpha e^{\gamma p}) \quad (97)$$

and (22) reduces to the particular case

$$VV'' = V'(V' + \alpha).$$

While a computation of positive conservation laws is based on expansion (19) at the vicinity $q \rightarrow \infty$, negative conservation laws can be found utilizing expansion (28) at the vicinity $q \rightarrow 0$. A generating function of conservation law densities

$$p = -\frac{1}{\gamma} \ln \left(\alpha + \frac{\gamma}{q(\lambda, \mathbf{A})} \right)$$

is an inverse expression to $q = U'(p)$ (see (97)). The first series of higher (positive) conservation law densities can be found by a substitution of an inverse expansion $q(\lambda, \mathbf{A})$ from (19) (see (91))

$$\lambda(q, \mathbf{A}) = \sigma q - 2\xi \ln q + (\alpha q + \gamma) \sum_{k=0}^{\infty} \frac{A^k}{q^k}, \quad q \rightarrow \infty.$$

The second series of lower (negative) conservation law densities can be found by a substitution of an inverse expansion $q(\lambda, \mathbf{A})$ from (28)

$$\lambda(q, \mathbf{A}) = (\alpha q^2 + \gamma q) \sum_{k=-\infty}^{-1} q^{-k-1} A^k - \sigma q, \quad q \rightarrow 0. \quad (98)$$

Then substitution of these series of conservation law densities into generating function of conservation laws (92)

$$p_{t^1} = \frac{\gamma b_1}{\alpha} \partial_x \ln \frac{\alpha A^0 + \sigma}{1 - \alpha e^{\gamma p}} \quad (99)$$

allows to extract both series of conservation laws (the above expression is obtained by virtue of (97)).

Remark: Expansion (98) is found by the same computation as in the previous Sections. The Vlasov type kinetic equation

$$\lambda_{t^1} = \gamma b_1 \left(U'(p) \lambda_x - \lambda_p \frac{A_x^0}{\alpha A^0 + \sigma} \right)$$

is connected with (99) by a semi-hodograph transformation $p(x, t; \lambda) \rightarrow \lambda(x, t; p)$. Taking into account the negative part of integrable hydrodynamic chain (94)

$$A_{t^1}^k = k(\alpha A^{k+1} + \gamma A^k) f'' A_x^0 + \gamma b_1 A_x^{k+1}, \quad k = -1, -2, \dots,$$

a substitution of expansion (28) into this Vlasov type kinetic equation leads to (98).

4.2 Generating function of commuting flows and conservation laws

An integrable hierarchy of commuting hydrodynamic chains described in this Section can be embedded into a sole generating function of conservation laws and commuting flows (33). The function $G(p(\lambda), p(\zeta))$ can be found from the compatibility condition of (33) and (92)

$$\partial_{t^1} p(\lambda) = \partial_x [u + U(p(\lambda))], \quad \partial_{\tau(\zeta)} p(\lambda) = \partial_x G(p(\lambda), p(\zeta)).$$

Theorem: *The function $G(p(\lambda), p(\zeta))$ is defined by the quadrature*

$$dG(p(\lambda), p(\zeta)) = \left(Q(p(\zeta)) - \frac{R(p(\zeta))}{U'(p(\zeta)) - U'(p(\lambda))} \right) dp(\lambda) + \frac{R(p(\zeta))}{U'(p(\zeta)) - U'(p(\lambda))} dp(\zeta),$$

where

$$R(p(\zeta)) = \exp[2\alpha U(p(\zeta)) + \gamma p(\zeta)], \quad Q'(p(\zeta)) = \alpha R(p(\zeta)) \quad (100)$$

Proof: The compatibility condition $\partial_{t^1}(\partial_{\tau(\zeta)} p(\lambda)) = \partial_{\tau(\zeta)}(\partial_{t^1} p(\lambda))$ implies

$$\partial_{\tau(\zeta)} u = Q(p(\zeta)) u_x + R(p(\zeta)) \partial_x p(\zeta), \quad (101)$$

$$\frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\lambda)} = Q(p(\zeta)) - \frac{R(p(\zeta))}{U'(p(\zeta)) - U'(p(\lambda))}, \quad \frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\zeta)} = \frac{R(p(\zeta))}{U'(p(\zeta)) - U'(p(\lambda))},$$

where functions $Q(p(\zeta))$ and $R(p(\zeta))$ are not yet determined. The compatibility condition

$$\frac{\partial}{\partial p(\lambda)} \frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\zeta)} = \frac{\partial}{\partial p(\zeta)} \frac{\partial G(p(\lambda), p(\zeta))}{\partial p(\lambda)}$$

leads to (100). Theorem is proved.

Moreover (101) reduces to (see the second equation in (100))

$$\partial_{\tau(\zeta)} e^{\alpha u} = (Q(p(\zeta)) e^{\alpha u})_x$$

It is easy to see the conservation law density $e^{\alpha u}$ is nothing but a momentum density A^0 (up to unessential additive and multiplicative constants).

4.3 Integrable three dimensional quasilinear equations of the second order

Compatibility conditions $(p_{t^k})_{t^n} = (p_{t^n})_{t^k}$ lead to integrable three dimensional hydrodynamic type systems. If $k = 1$ and $n = -1$, then three dimensional hydrodynamic type system

$$u_{t^{-1}} + v_{t^0} + \alpha v u_{t^0} = 0, \quad v_{t^1} = \gamma v u_{t^0} \quad (102)$$

is determined by the dispersionless Lax pair (see (96) and (92))

$$p_{t^1} = \partial_{t^0}(u + U(p)), \quad p_{t^{-1}} = \partial_{t^0} \left(\frac{v}{U'(p)} \right), \quad (103)$$

where

$$u = \frac{1}{\alpha} \ln(\alpha A^0 + \sigma), \quad v = \frac{\gamma}{\gamma A^{-1} - \sigma}.$$

Hydrodynamic type system (102) can be written in the conservative form

$$(e^{\alpha u})_{t^{-1}} + \alpha(v e^{\alpha u})_{t^0} = 0, \quad (\ln v)_{t^1} = \gamma u_{t^0}.$$

Introducing the potential function z such that (see the first conservation law)

$$u = \frac{1}{\alpha} \ln z_{t^0}, \quad v = -\frac{z_{t^{-1}}}{\alpha z_{t^0}},$$

(102) reduces to the three dimensional quasilinear equation of the second order

$$\alpha(z_{t^0} z_{t^{-1}t^1} - z_{t^{-1}} z_{t^0t^1}) = \gamma z_{t^{-1}} z_{t^0t^0};$$

introducing the potential function \tilde{z} such that (see the second conservation law)

$$v = e^{\tilde{z}_{t^0}}, \quad u = \frac{\tilde{z}_{t^1}}{\gamma},$$

(102) reduces to another three dimensional quasilinear equation of the second order

$$\tilde{z}_{t^1t^{-1}} + e^{\tilde{z}_{t^0}}(\gamma \tilde{z}_{t^0t^0} + \alpha \tilde{z}_{t^0t^1}) = 0. \quad (104)$$

A substitution (where the potential function ψ is introduced for (103), see (97))

$$u = \psi_{t^1} + \frac{1}{\alpha} \ln(1 - \alpha e^{\gamma \psi_{t^0}}), \quad v = \frac{\gamma e^{\gamma \psi_{t^0}}}{1 - \alpha e^{\gamma \psi_{t^0}}} \psi_{t^{-1}}$$

into (102) implies the integrable quasilinear three dimensional equation of the second order

$$\psi_{t^1t^{-1}} + \frac{\gamma e^{\gamma \psi_{t^0}}}{1 - \alpha e^{\gamma \psi_{t^0}}} \psi_{t^{-1}} (\alpha \psi_{t^0t^1} + \gamma \psi_{t^0t^0}) = 0.$$

If $k = 1$ and $n = 2$, then the dispersionless Lax pair (see (81) and (99))

$$p_{t^1} = \partial_{t^0}(u + U(p)), \quad p_{t^2} = \partial_{t^0}(w + sU'(p)) \quad (105)$$

determines the constraint $s = e^{\alpha u}$ and the three dimensional hydrodynamic type system written in the conservative form

$$\alpha(e^{\alpha u})_{t^1} = (\alpha w - \gamma e^{\alpha u})_{t^0}, \quad u_{t^2} = w_{t^1}. \quad (106)$$

Introducing the potential function z such that (see the first conservation law)

$$u = \frac{1}{\alpha} \ln z_{t^0}, \quad w = z_{t^1} + \frac{\gamma}{\alpha} z_{t^0},$$

(106) reduces to the three dimensional quasilinear equation of the second order

$$z_{t^0t^2} = z_{t^0}(\alpha z_{t^1t^1} + \gamma z_{t^0t^1});$$

introducing the potential function \bar{z} such that (see the second conservation law)

$$u = \bar{z}_{t^1}, \quad w = \bar{z}_{t^2},$$

(106) reduces to another three dimensional quasilinear equation of the second order

$$\bar{z}_{t^0 t^2} = e^{\alpha \bar{z}_{t^1}} (\alpha \bar{z}_{t^1 t^1} + \gamma \bar{z}_{t^0 t^1}). \quad (107)$$

A substitution (where the potential function ψ is introduced for (105), see (97))

$$u = \psi_{t^1} + \frac{1}{\alpha} \ln(1 - \alpha e^{\gamma \psi_{t^0}}), \quad w = \psi_{t^2} - \gamma e^{\gamma \psi_{t^0} + \alpha \psi_{t^1}}$$

into (106) implies the integrable quasilinear three dimensional equation of the second order

$$\psi_{t^0 t^2} = e^{\alpha \psi_{t^1}} (1 - \alpha e^{\gamma \psi_{t^0}}) (\gamma \psi_{t^0 t^1} + \alpha \psi_{t^1 t^1}).$$

Remark: The whole hierarchy of positive and negative commuting flows possesses the symmetry $t^{k+1} \leftrightarrow t^{-k}$. Without loss of generality it is enough to prove for the independent variables $t^1 \leftrightarrow t^0$ only. Indeed, first generating function of conservation laws (92)

$$p_t = \partial_x(u + U(p))$$

can be rewritten in the same form

$$q_x = \partial_t(v + \tilde{U}(q)),$$

where $q = U(p)$, $p = \tilde{U}(q)$, $v_t = -u_x$ (see (95)). Since (see (78) where $\epsilon = 0$ and (97)) $U''(p) = \alpha U'^2(p) + \gamma U'(p)$, then $\tilde{U}''(q) = -\gamma \tilde{U}'^2(q) - \alpha \tilde{U}'(q)$. Thus, equations (104) and (107) are equivalent to each other (by an appropriate change of the constants $\alpha \leftrightarrow \gamma$ and the independent variables $t^1 \leftrightarrow t^0$, $t^2 \leftrightarrow t^{-1}$).

Let us consider the three dimensional hydrodynamic type system written in the conservative form (see (99), (101) and (97))

$$\partial_\tau e^{\alpha u} = \partial_{t^0} \frac{\gamma e^{\alpha u}}{1 - \alpha e^{\gamma p}}, \quad \partial_{t^1} p = \partial_{t^0} \left(u - \frac{1}{\alpha} \ln(1 - \alpha e^{\gamma p}) \right). \quad (108)$$

(108) reduces to the integrable three dimensional quasilinear equation of the second order

$$\alpha(z_{t^0} z_{\tau t^1} - z_\tau z_{t^0 t^1}) = (z_\tau - \gamma z_{t^0}) z_{\tau t^0},$$

where the function z is a potential of the first conservation law; (108) reduces to another integrable three dimensional quasilinear equation of the second order

$$\psi_{\tau t^1} - \gamma \psi_{t^0 t^1} = e^{\gamma \psi_{t^0}} (\gamma \psi_{\tau t^0} + \alpha \psi_{\tau t^1}),$$

where the function ψ is a potential of the second conservation law.

5 Conclusion

In this paper, integrable Hamiltonian hydrodynamic chains associated with Dorfman Poisson brackets are described. These Dorfman Poisson brackets are parameterized by six $(\alpha, \beta, \gamma, \delta, \epsilon, \xi)$ constants. However, three distinguish cases are selected by special values of two constants β and δ only. The first and most general class is determined by arbitrary parameters $(\alpha \neq 0, \beta \neq 0, \gamma \neq 0, \delta \neq 0)$; the second class is determined by the restriction $\beta = 0$ while $\delta \neq 0$; the third class is determined by more deep restriction $\beta = 0$ and $\delta = 0$. In the particular case $\delta = 0$, hydrodynamic chains of the first class were constructed in [27]; corresponding Hamiltonian reductions are the Hamiltonian chromatography systems. Hydrodynamic chains of the second class also are well known. In a particular case (a special limit $\alpha = 0$), they are nothing but the Kupershmidt hydrodynamic chains (see [3], [17], [21] and [26]). The most investigated Benney hydrodynamic chain (see [1], [12], [13], [14], [15], [16], [19], [32]) belongs to the third class.

In a general case, hydrodynamic chains of the third class are considered in [28]. However, the simplest hydrodynamic chain (82) was missed. This hydrodynamic chain is found (as well as corresponding hydrodynamic chain (53) of the second class) in the presented paper. Corresponding Hamiltonian density depends on *infinitely many* field variables (moments A^k). This phenomenon was never mentioned in a literature.

All integrable hydrodynamic chains possess infinite series of conservation laws and commuting flows. Corresponding generating functions are constructed. These hydrodynamic chains are extended on negative values of moments A^k as well as associated integrable hierarchies are extended on negative values of time variables t^n .

Most obvious and simple three dimensional hydrodynamic type systems are presented. They are converted to the three dimensional quasilinear equations of the second order (see a general theory in [4]). Most of them are new. Nevertheless, the list of these equations (as well as associated hydrodynamic type systems) is so large, that all already found examples in [10] can be extracted from this list by virtue of different parametric reductions. Moreover, the list presented in [10] can be split on three parts according given approach here. Examples (18) and (21) belonging to the Benney hierarchy are of the third class; examples (20)₁, (20)₂ and (22) belong to the Kupershmidt hierarchy (see [26], where $\beta = 3$); most complicated examples (16), (17) and (19) belong to the first class, where $V(p) = \wp(p)$ and the elliptic Weiersstrass function satisfies $\wp'^2(p) = 4\wp^3(p) + \bar{\delta}^2$, while the function $V(p)$ satisfies $2V(p)V''(p) = 3(V'^2(p) - \bar{\delta}^2)$ (see (22)). This is a full list of integrable three dimensional Hamiltonian hydrodynamic type systems (see detail in [10])

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_y \end{pmatrix} \begin{pmatrix} h_u \\ h_v \end{pmatrix}.$$

In another paper [11], a full list of integrable three dimensional Hamiltonian hydrodynamic type systems

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & \partial_x \\ \partial_x & \partial_y \end{pmatrix} \begin{pmatrix} h_u \\ h_v \end{pmatrix}$$

also is presented. Examples (21) and (24) belong again to the Benney hierarchy; examples (20), (22)₁ and (23) belong to the Kupershmidt hierarchy (see [26], where $\beta = 3$); example

(22)₂ belongs to the first class, where $V(p) = \wp(p)$. And just the last example (19) is not yet recognized. Thus, most these examples associated with two dimensional constant Poisson brackets simultaneously are connected with hydrodynamic chains associated with Dorfman Poisson brackets (but not vice versa!). However, the question “how to connect both Hamiltonian structures” is open at this moment.

The relationship between three dimensional hydrodynamic type systems, three dimensional quasilinear equations of the second order and hydrodynamic chains described above is very important. All hydrodynamic chains (2) presented in this paper are very convenient for more deep investigation by the moment decomposition approach (see detail in [29] and in [24]). This approach allows to extract multi-parametric solutions (see detail in [30]) of these hydrodynamic chains and corresponding three dimensional hydrodynamic type systems as well as three dimensional quasilinear equations of the second order. Thus, any aforementioned example can be equipped by a corresponding hydrodynamic chain described in this paper. It means, that such a hydrodynamic chain can be considered as an infinite set of the so called pseudo-nonlocalities (i.e. moments) allowing to extend integrable three dimensional two component hydrodynamic type systems on infinitely many field variables. It means that a complexity of integrable three dimensional quasilinear equations of the second order can be translated to a complexity of two dimensional hydrodynamic chains (i.e. hydrodynamic type systems containing infinitely many equations).

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